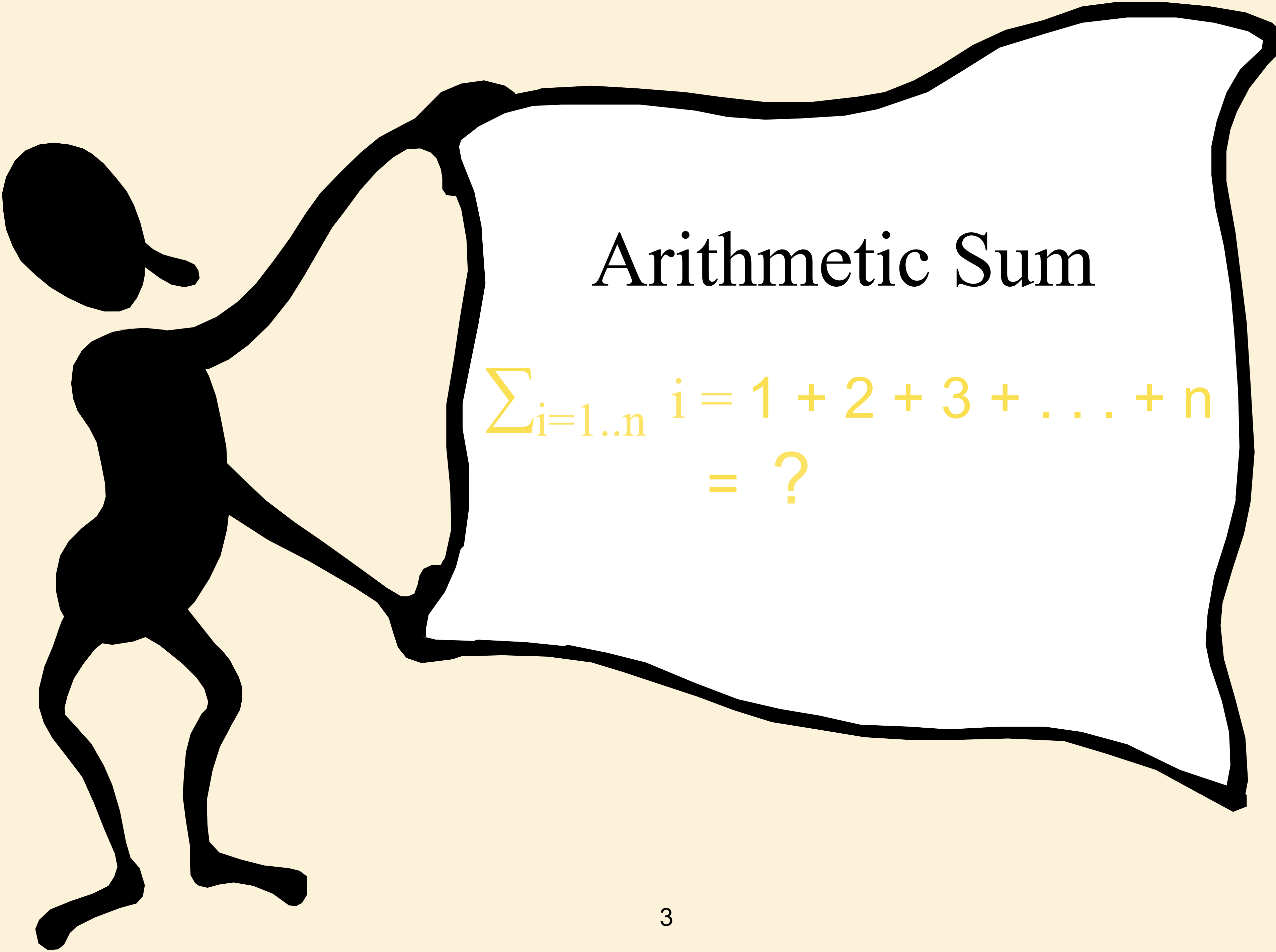
# Summations

Evaluating $\sum_{i=1}^{n} f(i)$

# Recall: Insertion Sort

| INSERTION-SORT($A$) | *cost* | *times* |
| --- | --- | --- |
| **for** $j \leftarrow 2$ **to** *length*$[A]$ | $c_1$ | $n$ |
| **do** *key* $\leftarrow A[j]$ | $c_2$ | $n-1$ |
| ▷ Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$. | $0$ | $n-1$ |
| $i \leftarrow j-1$ | $c_4$ | $n-1$ |
| **while** $i > 0$ and $A[i] > key$ | $c_5$ | $\sum_{j=2}^{n} t_j$ |
| **do** $A[i+1] \leftarrow A[i]$ | $c_6$ | $\sum_{j=2}^{n} (t_j - 1)$ |
| $i \leftarrow i-1$ | $c_7$ | $\sum_{j=2}^{n} (t_j - 1)$ |
| $A[i+1] \leftarrow key$ | $c_8$ | $n-1$ |

Worst case (reverse order): $t_j = j$: $\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \rightarrow T(n) \in \theta(n^2)$


Arithmetic Sum
$\sum_{i=1..n} i = 1 + 2 + 3 + . . . + n$
$= ?$

$$1 + 2 + 3 + \ldots + n-1 + n = S$$

$$n + n-1 + n-2 + \ldots + 2 + 1 = S$$

---

$$(n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) = 2S$$

$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + \dots + n-1 + n = S$$

$$n + n-1 + n-2 + \dots + 2 + 1 = S$$

$$(n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) = 2S$$

$$n(n+1) = 2S$$

**Algebraic argument**

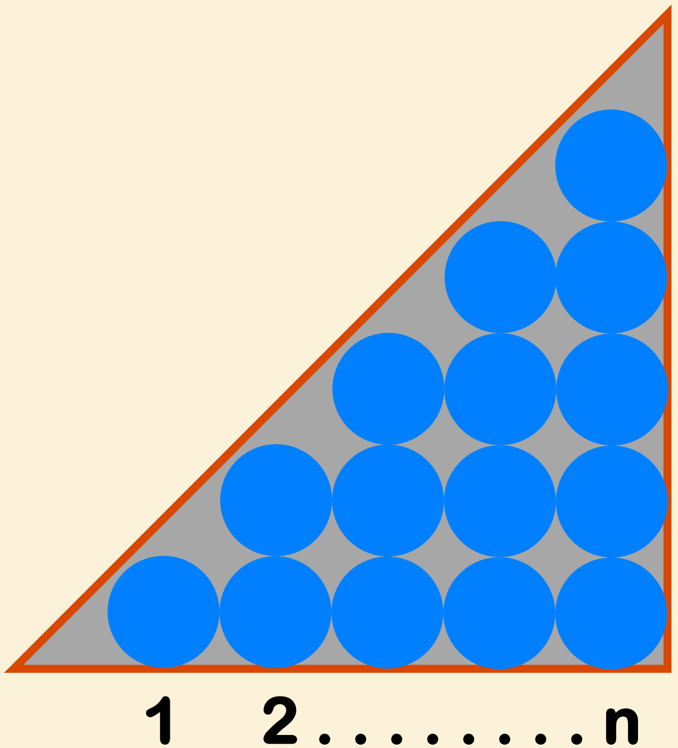
# Let's restate this argument using a <u>geometric</u> representation

$1 + 2 + 3 + \ldots + n-1 + n = S$ = number of blue dots.

$n + n-1 + n-2 + \ldots + 2 + 1 = S$

$(n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) = 2S$

$n\,(n+1) = 2S$

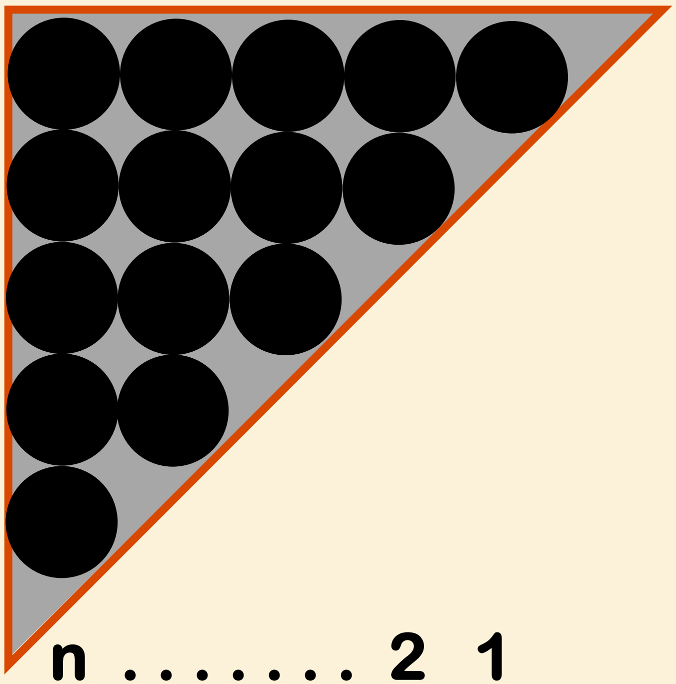


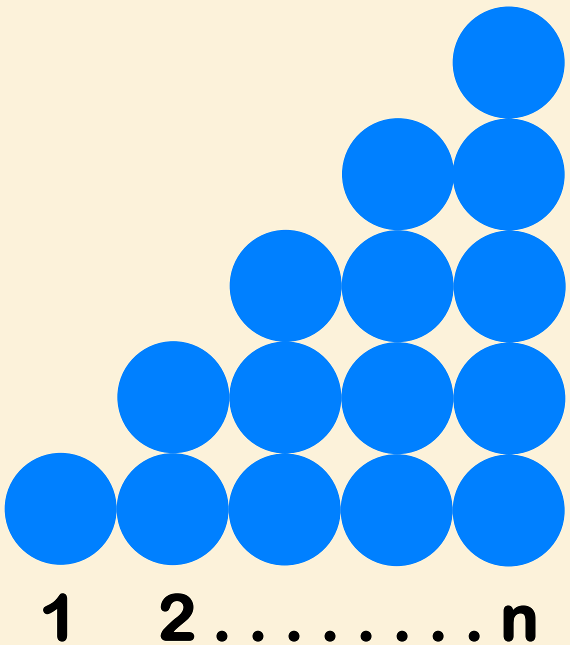
$1 + 2 + 3 + \dots + n-1 + n = S$ = number of blue dots

$n + n-1 + n-2 + \dots + 2 + 1 = S$ = number of black dots

$(n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) = 2S$

$n(n+1) = 2S$



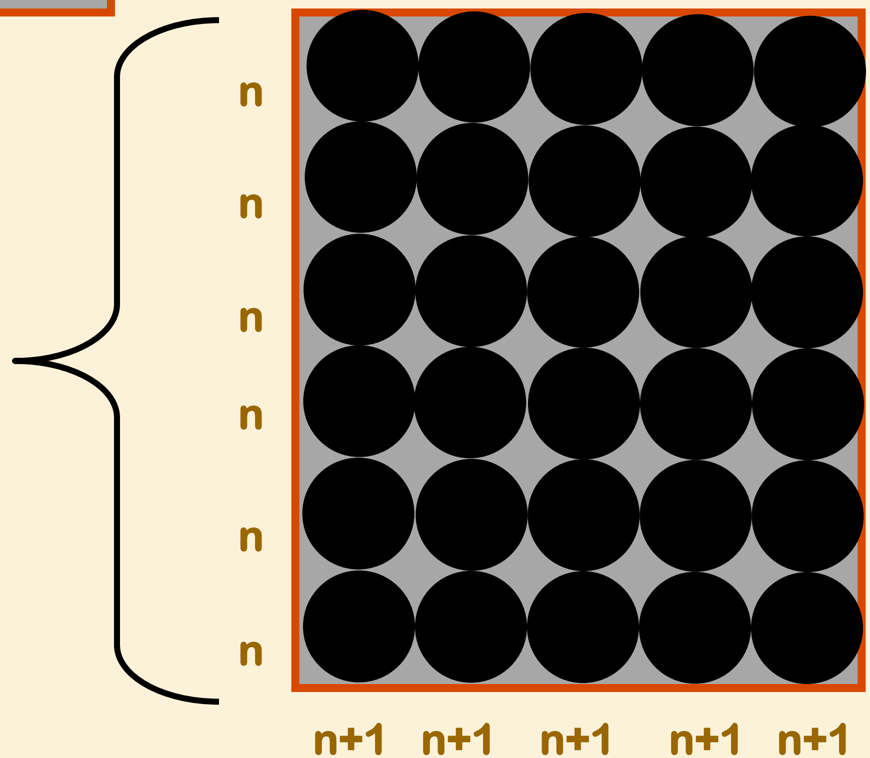
$$1 + 2 + 3 + \ldots + n-1 + n = S$$ = number of blue dots

$$n + n-1 + n-2 + \ldots + 2 + 1 = S$$ = number of black dots

$$(n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) = 2S$$

$$n(n+1) = 2S$$

There are n(n+1) dots in the grid



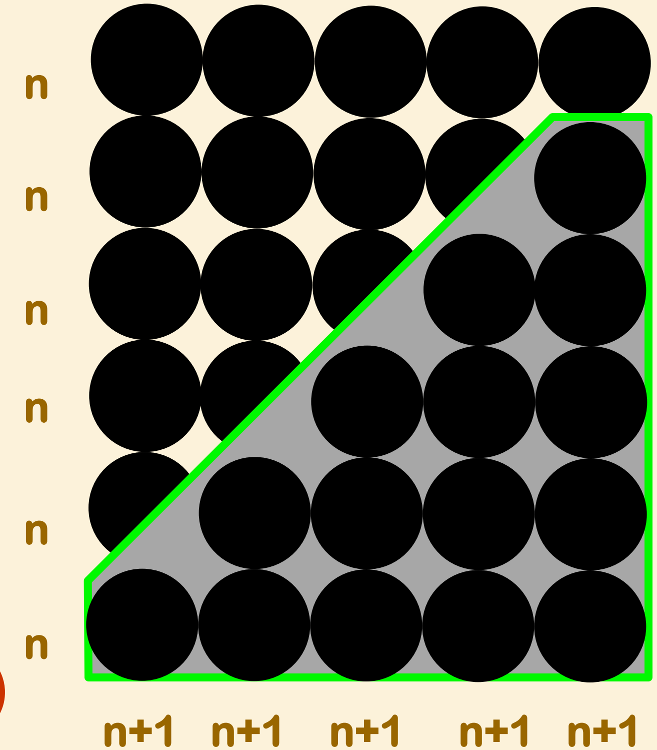
1 + 2 + 3 + . . . + n-1 + n = S = number of blue dots

n + n-1 + n-2 + . . . + 2 + 1 = S = number of black dots

(n+1) + (n+1) + (n+1) + . . . + (n+1) + (n+1) = 2S

n (n+1) = 2S

$$S = \frac{n\ (n + 1)}{2}$$




Note = $\Theta$(# of terms · last term))

# Arithmetic Sum

$$\sum_{i=1..n} i = 1 + 2 + 3 + \ldots + n$$

$$= \Theta(\#\text{ of terms} \cdot \text{last term})$$

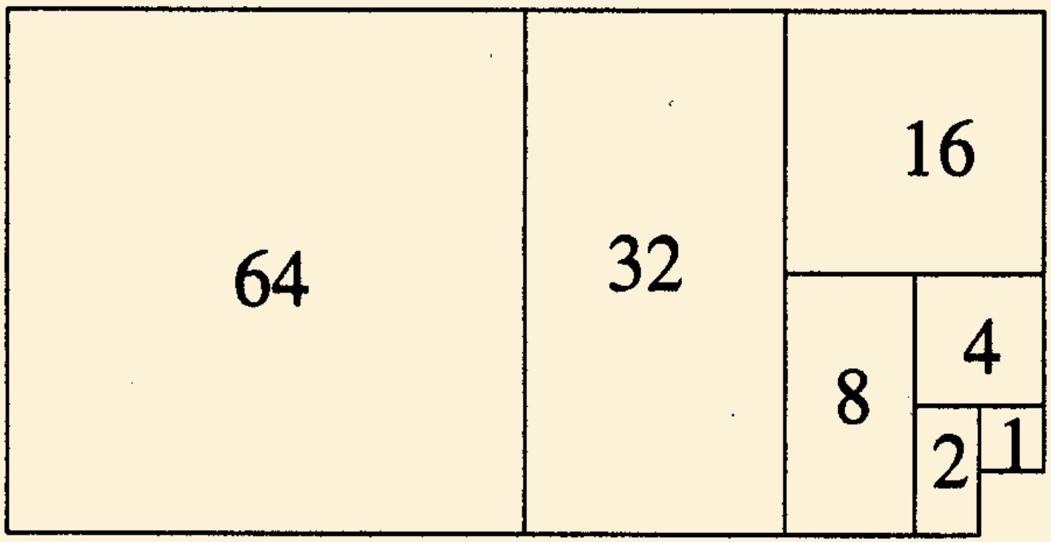
True whenever terms increase slowly

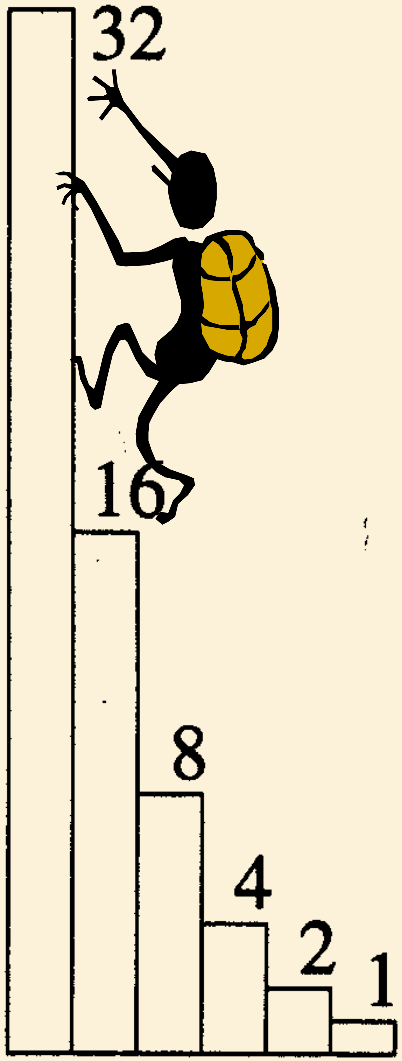
# Geometric Sum

$\sum_{i=0..n} 2^i = 1 + 2 + 4 + 8 + . . . + 2^n$

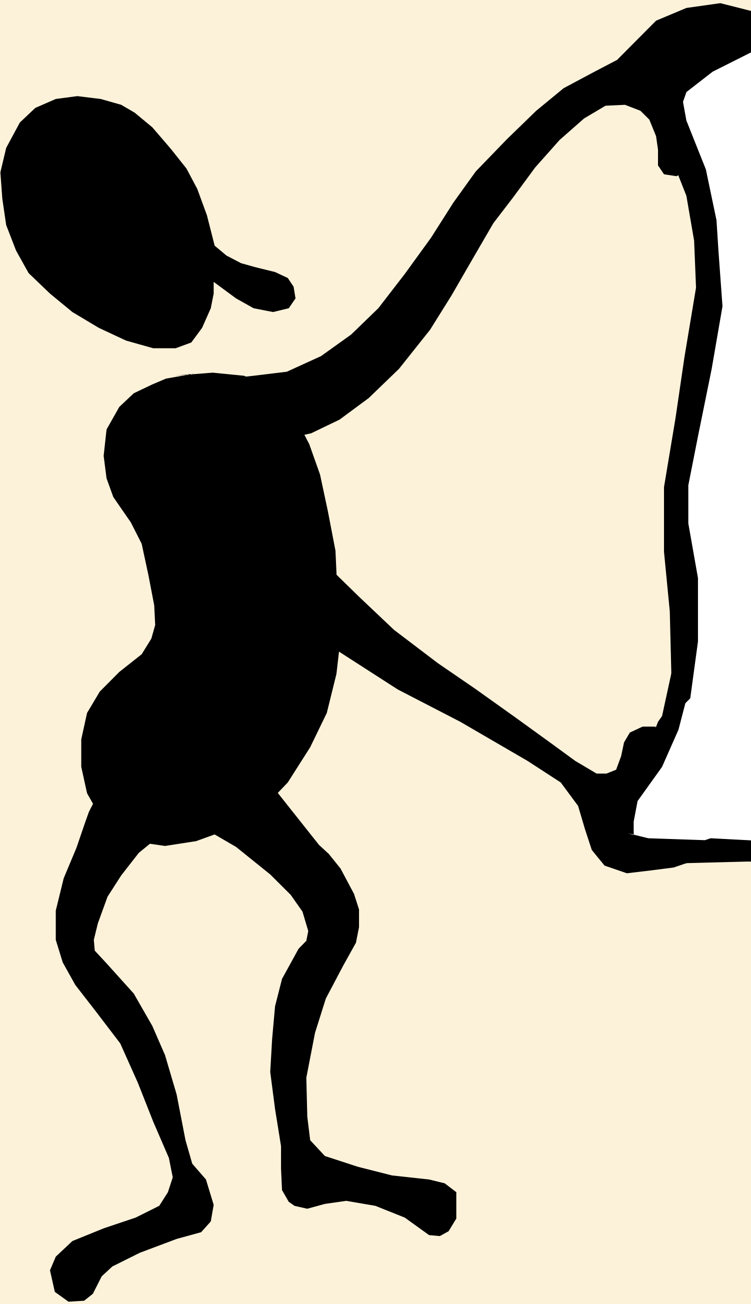
$= ?$

# Geometric Sum




1+2+4+8+16+32+64 = 2*64 - 1



# Geometric Sum

$$\sum_{i=0..n} 2^i = 1 + 2 + 4 + 8 + . . . + 2^n$$
$$= 2 \cdot \text{last term} - 1$$

# Geometric Sum

$$\sum_{i=0..n} r^i = r^0 + r^1 + r^2 + . . . + r^n$$

$$= \ ?$$

# Geometric Sum

$$S = 1 + r + r^2 + r^3 + \dots + r^n$$

$$Sr = r + r^2 + r^3 + \dots + r^n + r^{n+1}$$

$$S - Sr = 1 - r^{n+1}$$

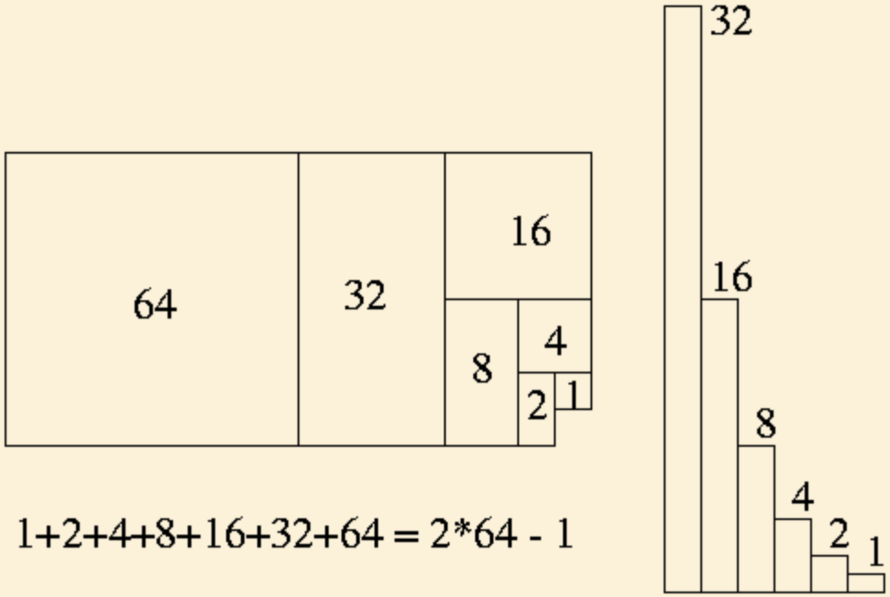
$$S = \frac{r^{n+1} - 1}{r - 1}$$

# Geometric Sum

$$\sum_{i=0..n} r^i = \frac{r^{n+1} - 1}{r - 1} = \theta(r^n)$$

When $r>1$

**Biggest Term**



# Geometric Increasing

$\sum_{i=0..n} r^i = r^0 + r^1 + r^2 + . . . + r^n$

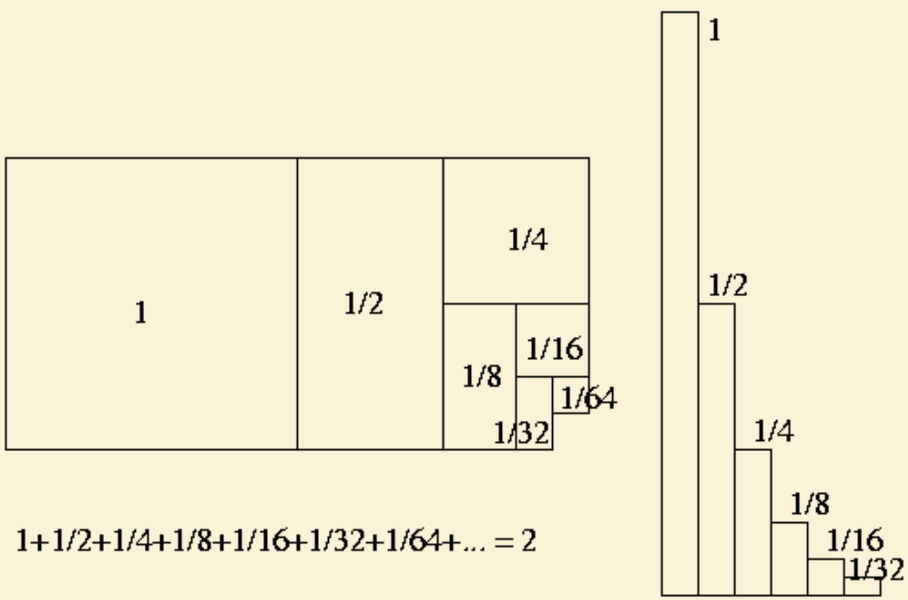
$= \Theta(\text{biggest term})$

True whenever terms increase quickly

# Geometric Sum

$$\sum_{i=0..n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

When $r<1$?

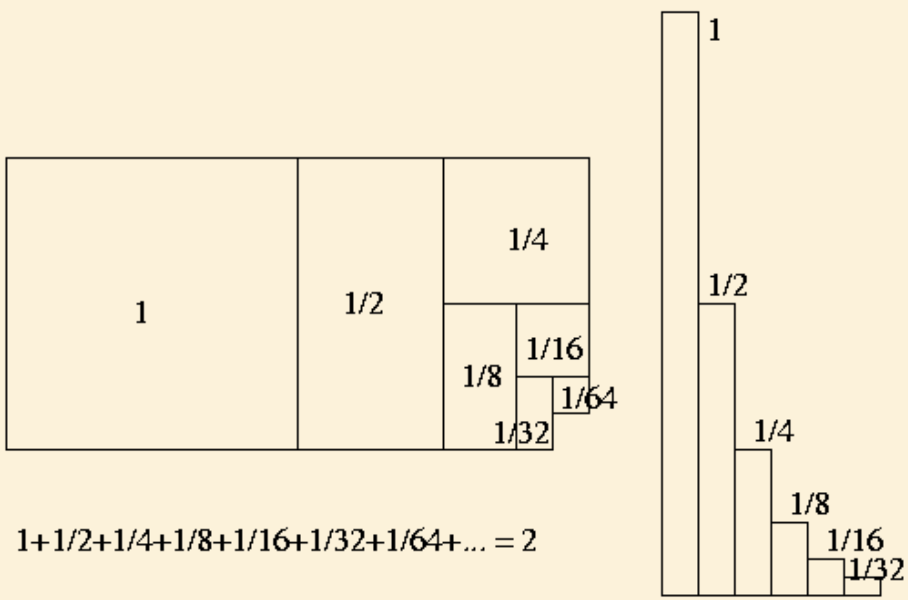


# Geometric Sum

$$\sum_{i=0..n} r^i = \frac{1 - r^{n+1}}{1 - r} = \theta(1)$$

When $r<1$

**Biggest Term**



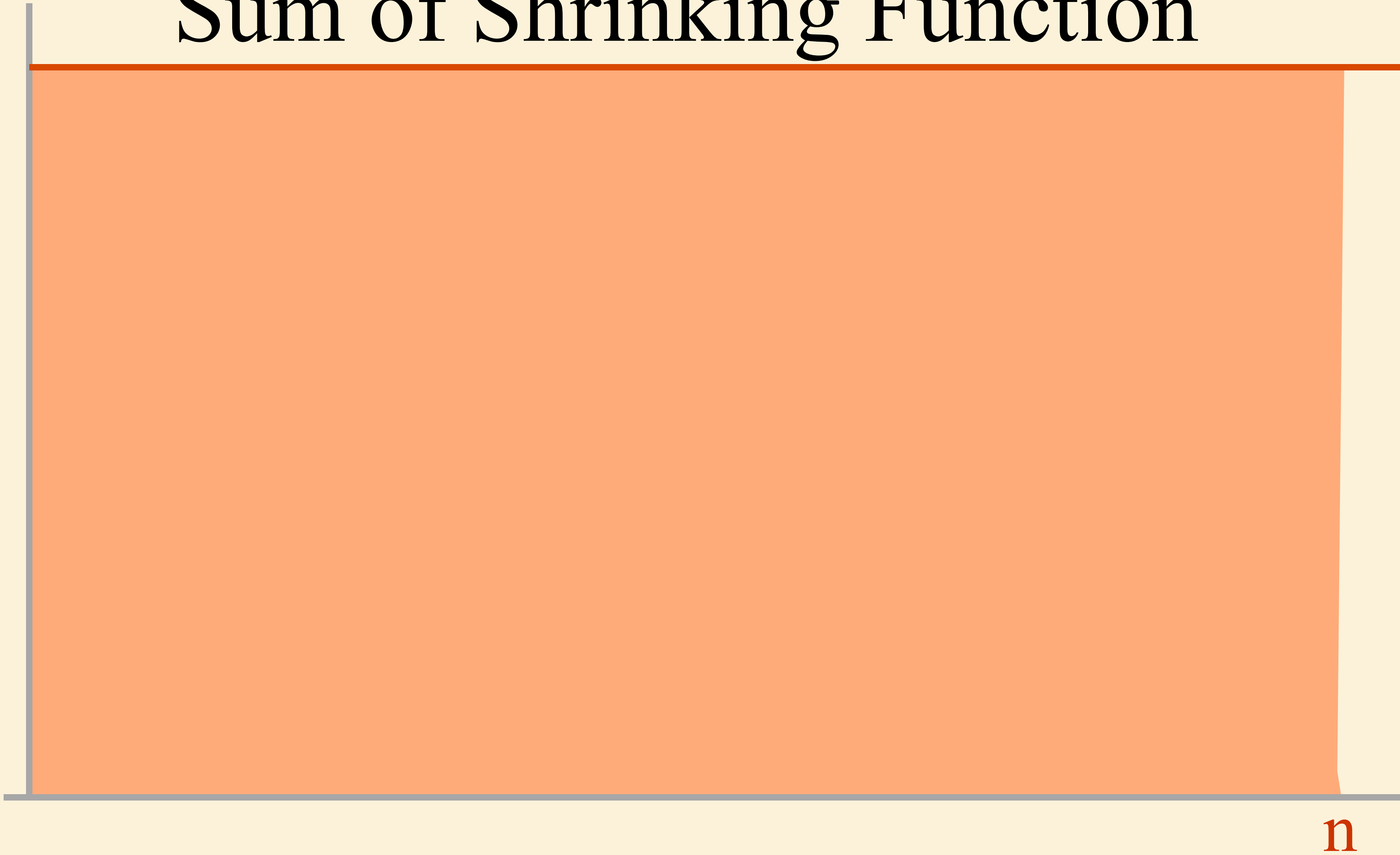
# Bounded Tail

$$\sum_{i=0..n} r^i = r^0 + r^1 + r^2 + \ldots + r^n$$

$$= \Theta(1)$$

True whenever terms decrease quickly

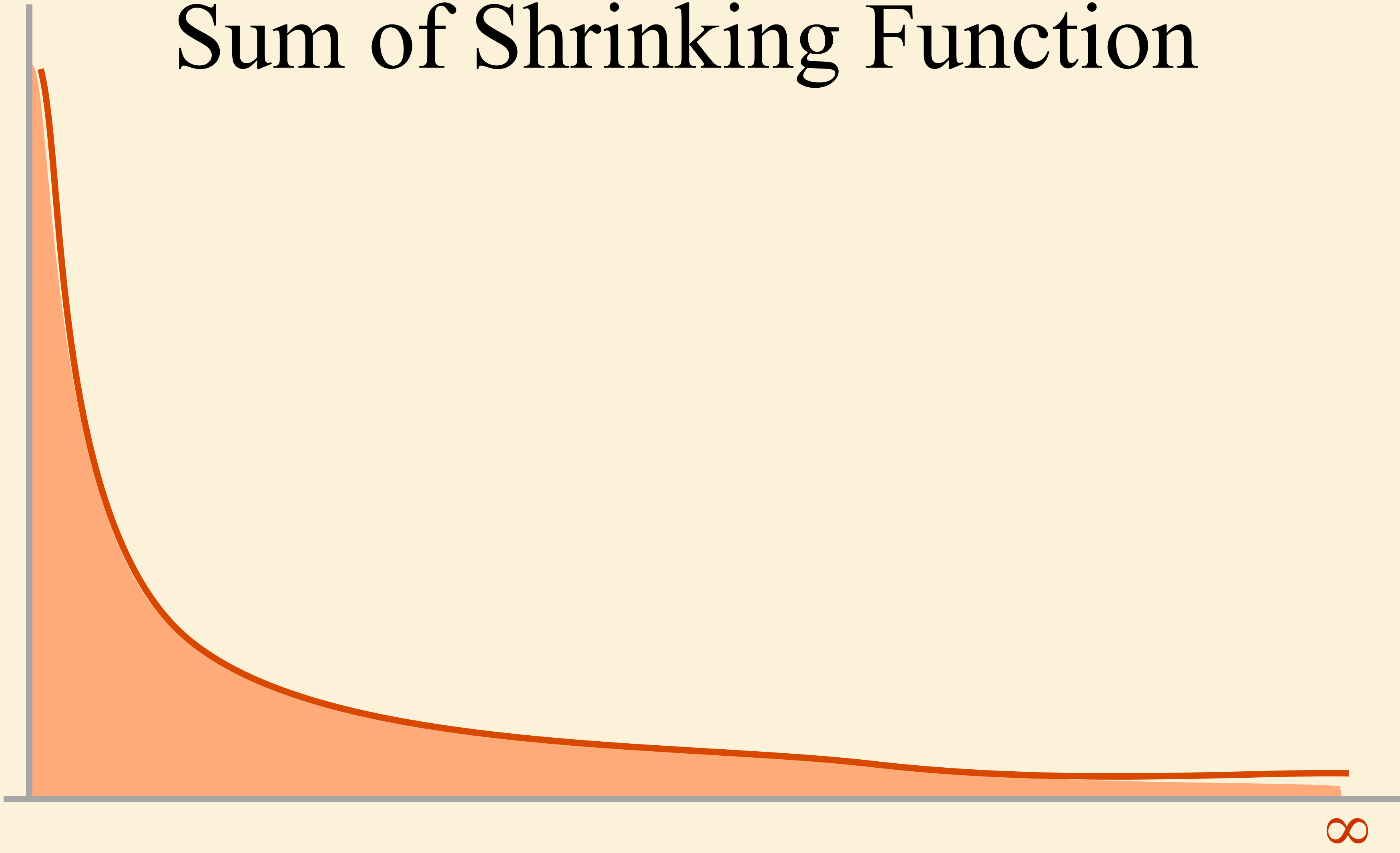
# Sum of Shrinking Function




$f(i) = 1$

$\sum_{i=1..n} f(i) = n$

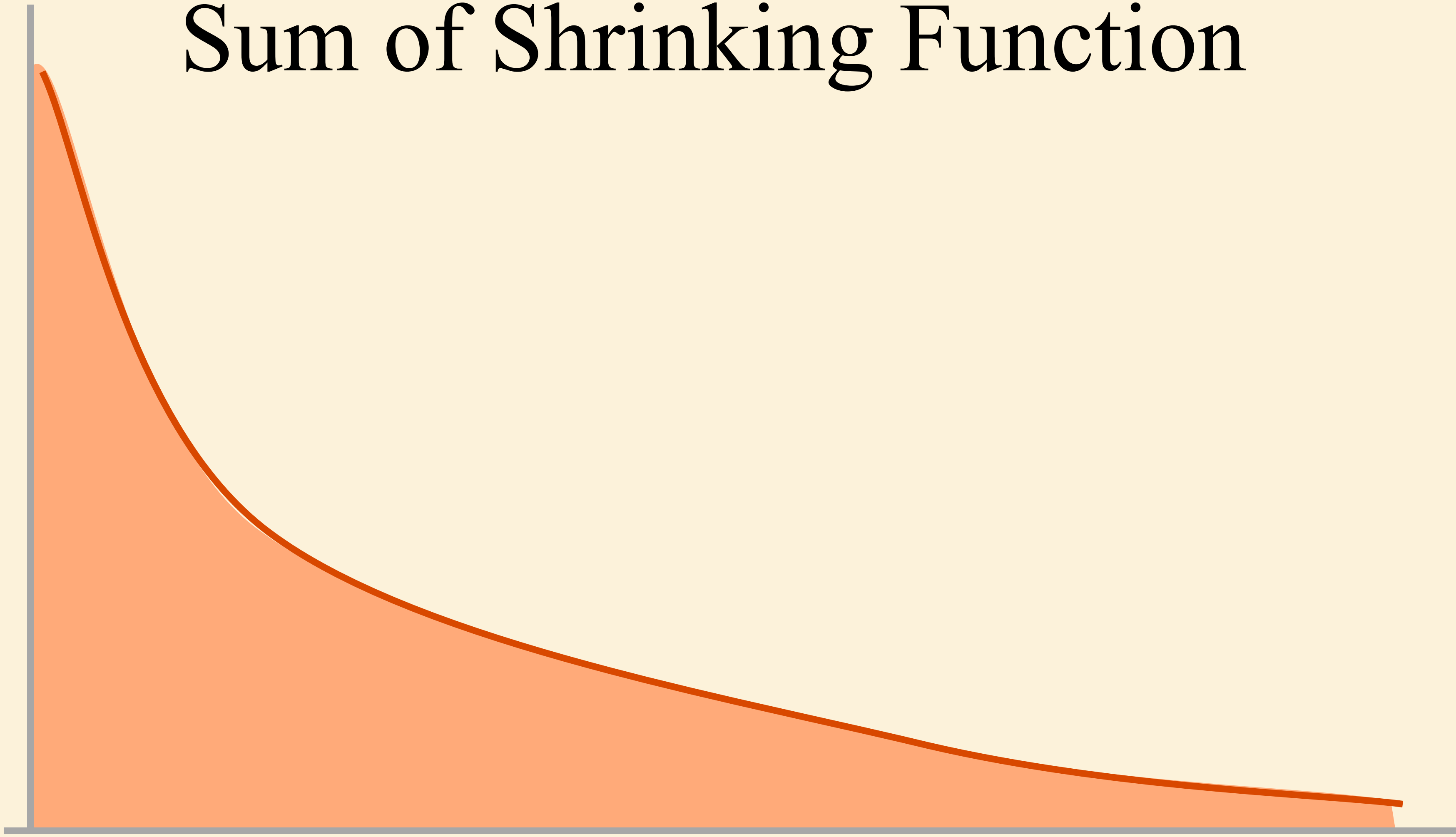
# Sum of Shrinking Function




$f(i) = 1/2^i$

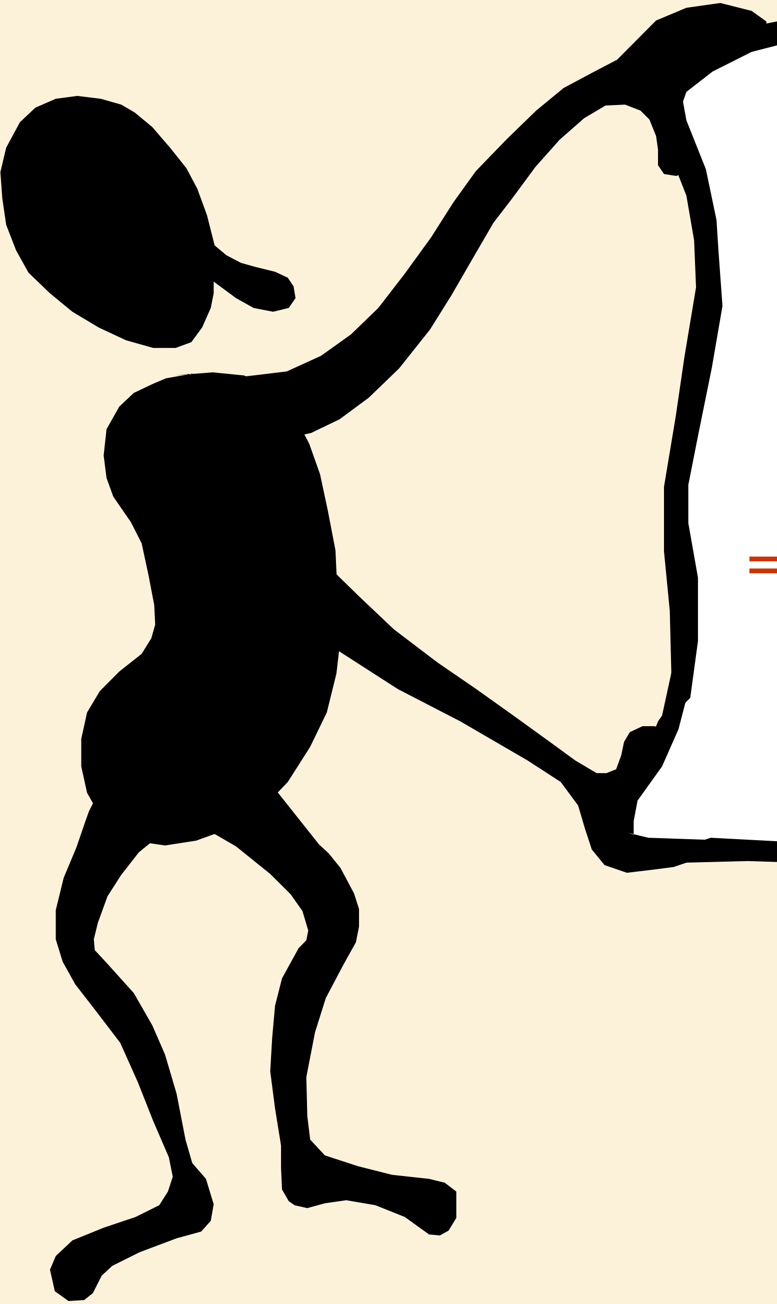
$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} \in \theta(1)$$

# Sum of Shrinking Function




$f(i) = 1/i$

$\sum_{i=1..n} f(i) = ?$



# Harmonic Sum

$$\sum_{i=1..n} {}^{1}/_{i}$$

$$= {}^{1}/_{1} + {}^{1}/_{2} + {}^{1}/_{3} + {}^{1}/_{4} + {}^{1}/_{5} + \ldots + {}^{1}/_{n}$$

$$= ?$$

# Harmonic Sum

$$\sum_{i=1}^{n}\frac{1}{i} = \underbrace{\overbrace{\frac{1}{1}}^{\geq 1\cdot\frac{1}{2}=\frac{1}{2}}}_{\leq 1\cdot 1=1} + \underbrace{\overbrace{\frac{1}{2}+\frac{1}{3}}^{\geq 2\cdot\frac{1}{4}=\frac{1}{2}}}_{\leq 2\cdot\frac{1}{2}=1} + \underbrace{\overbrace{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}^{\geq 4\cdot\frac{1}{8}=\frac{1}{2}}}_{\leq 4\cdot\frac{1}{4}=1} + \underbrace{\overbrace{\frac{1}{8}+\ldots+\frac{1}{15}}^{\geq 8\cdot\frac{1}{16}=\frac{1}{2}}}_{\leq 8\cdot\frac{1}{8}=1} + \ldots$$

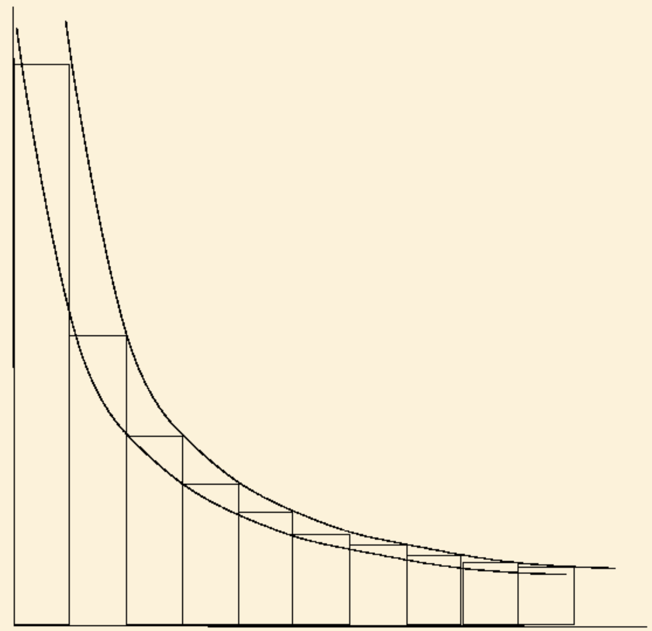
From this it follows that

$$\frac{1}{2}\lfloor \log_2 n \rfloor \leq \sum_{i=1}^{n}\frac{1}{i} \leq \lfloor \log_2(n+1) \rfloor + 1 \Rightarrow \sum_{i=1}^{n}\frac{1}{i} = \theta(\log_2 n)$$

# Harmonic Sum

$$\sum_{i=1..n} {}^{1}/_{i}$$
$$= {}^{1}/_{1} + {}^{1}/_{2} + {}^{1}/_{3} + {}^{1}/_{4} + {}^{1}/_{5} + \ldots + {}^{1}/_{n}$$
$$= \Theta(\log(n))$$

# Approximating Sum by Integrating



$$\sum_{i=1..n} f(i) \approx \int_{x=1..n} f(x)\, dx$$

The area under the curve approximates the sum

# Approximating Sums by Integrating: Harmonic Sum

$$\sum_{i=1}^{n} \frac{1}{i}\ ; \ \int_{1}^{n} \frac{1}{x}\, dx = \log x \Big|_{1}^{n} = \log n$$

# Approximating Sums by Integrating: Arithmetic Sums

$$\sum_{i=0}^{n} i^c \cong \int_0^n x^c dx = \frac{1}{c+1} n^{c+1} = \theta(n^{c+1}) = \theta(n \cdot n^c)$$

$= \theta(\text{number of terms} \times \text{last term})$

(True whenever terms increase slowly)

# Approximating Sums by Integrating: Geometric Sum

$$\sum_{i=0}^{n} b^i \cong \int_0^n b^x dx = \frac{1}{\ln b} b^x \Big|_0^n = \frac{1}{\ln b}(b^n - 1) = \theta(b^n)$$

$$= \theta(\text{last term})$$

(True whenever terms increase rapidly)

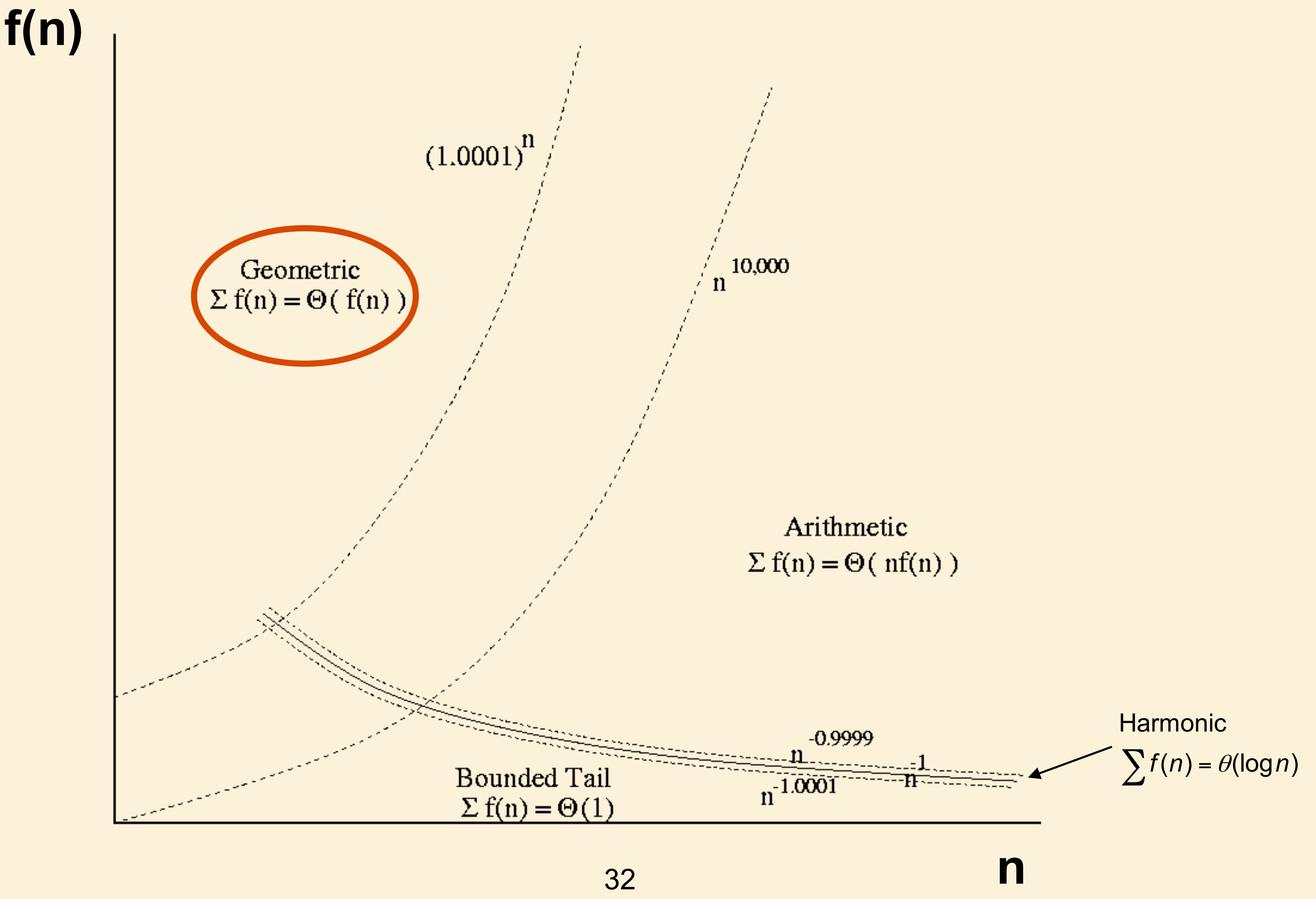
# Adding Made Easy

We will now classify (most) functions $f(i)$ into four classes:

- Geometric Like
- Arithmetic Like
- Harmonic
- Bounded Tail

For each class, we will give an easy rule for approximating its sum $\theta( \sum_{i=1..n} f(i) )$

# Adding Made Easy



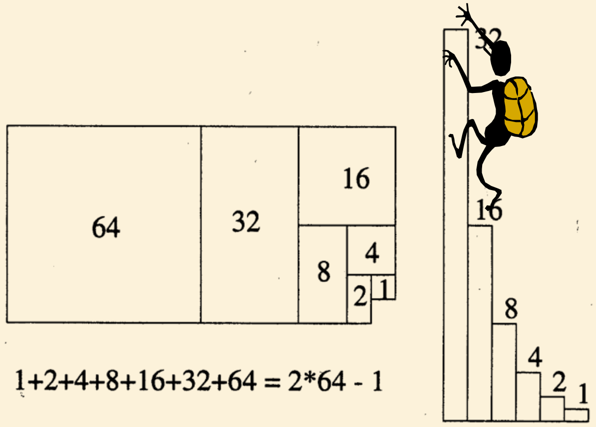
# Geometric Like:

$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=0..n} f(i) = \theta(f(n))$

If the terms f(i) grow sufficiently quickly, then the sum will be dominated by the largest term.

Classic example:

$\sum_{i=0..n} 2^i = 2^{n+1}-1 \approx 2\ f(n)$



# Geometric Like:

$$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$$

If the terms f(i) grow sufficiently quickly, then the sum will be dominated by the largest term.

For which functions f(i) is this true?

How fast and how slow can it grow?

# Geometric Like:

$$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$$

$$\sum_{i=1..n} (1000)^i \approx 1.001(1000)^n$$
$$= 1.001\ f(n)$$

Even bigger? 

# Geometric Like:

$$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$$

No Upper Extreme: $\sum_{i=1..n} 2^{2^{2^i}} \approx 2^{2^{2^n}} = 1 f(n)$

Even bigger!

# Geometric Like:

$$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$$

$$\text{e.g. } f(n) = 8\frac{2^n}{n^{100}} + n^3 = \theta(f(n)) = \theta(\frac{2^n}{n^{100}}) \in 2^{\theta(n)}$$

(The strongest function determines the class.)

# Geometric Like:

## $f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$

In general, if $f(n) = c_1 b^{c_2 n} n^{c_3} \log^{c_4} n$

where

$b > 1$

$c_1 \in \circ^{+}$

$c_2 \in \circ^{+}$

$c_3, c_4 \in \circ$

Then $f(n) \in 2^{\Omega(n)}$

and $\sum_{i=1}^{n} f(i) \in \Theta(f(n))$

# Geometric Like:

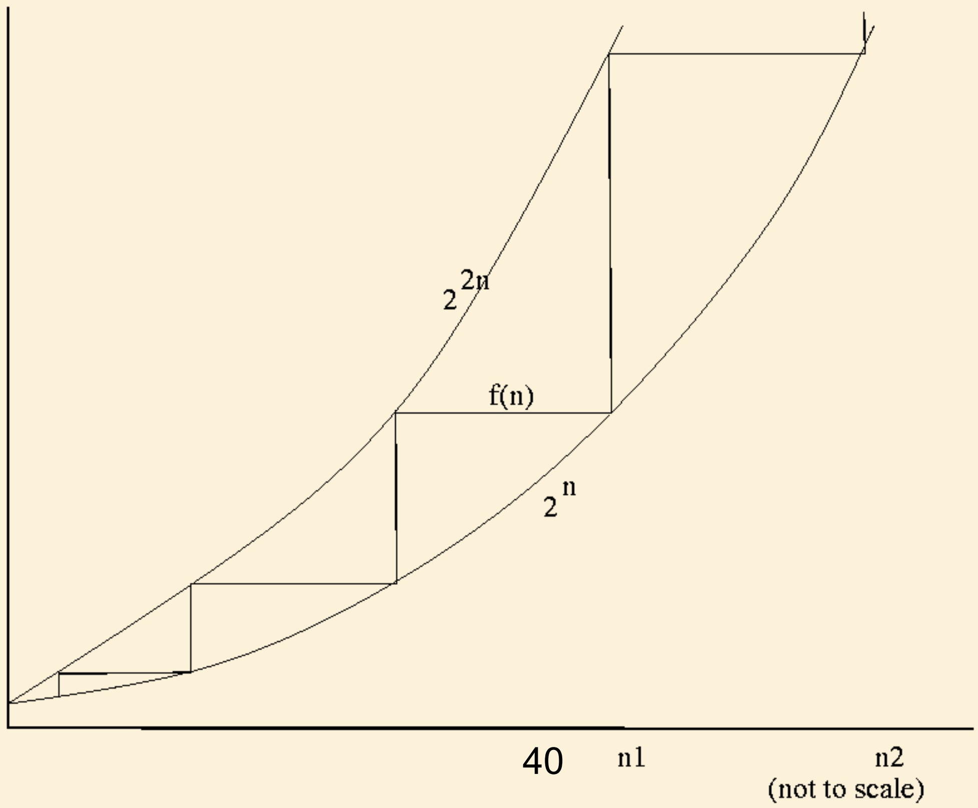
$$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$$

Do All functions in $2^{\Omega(n)}$ have this property?
Maybe not.

# Geometric Like:

$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$

Functions that oscillate with exponentially increasing amplitude do not have this property.



# Geometric Like:

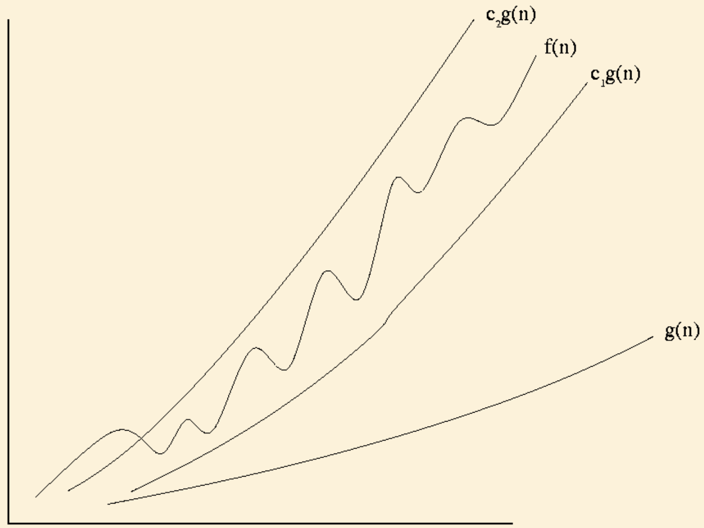
$f(n) \geq 2^{\Omega(n)} \Rightarrow \sum_{i=1..n} f(i) = \theta(f(n))$

Functions expressed with +, -, ·, ÷ , exp, log

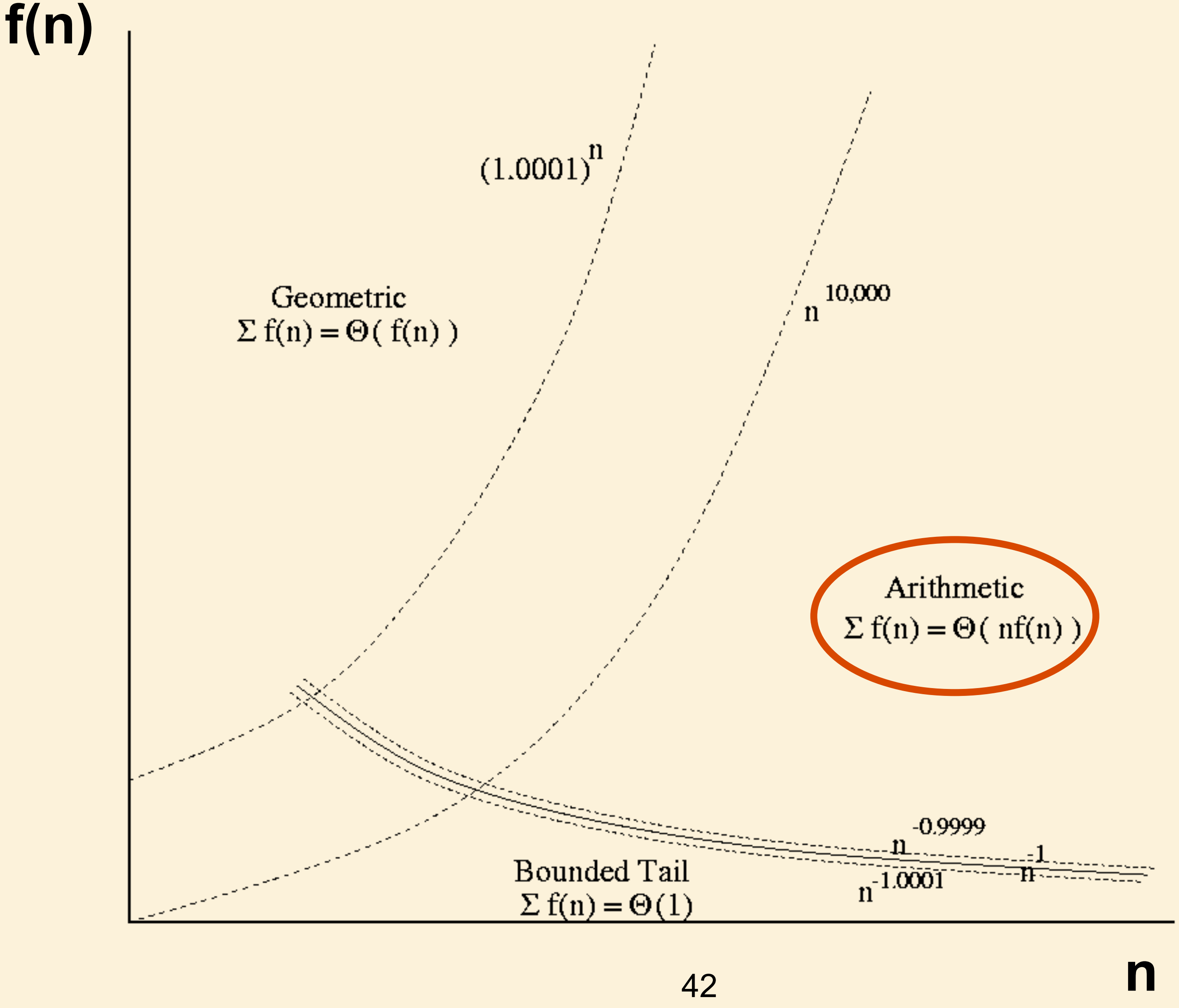
do not oscillate continually.

They are well behaved for sufficiently large n.

These do have this property.



# Adding Made Easy



# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

If the terms f(i) are increasing or decreasing relatively slowly, then the sum is roughly the number of terms, n, times the final value.

Example 1:

$$\sum_{i=1..n} 1 = n \cdot 1$$

# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

If the terms f(i) are increasing or decreasing relatively slowly, then the sum is roughly the number of terms, n, times the final value.

Example 2:

$$\sum_{i=1..n} i = 1 + 2 + 3 + \ldots + n$$

$$= \frac{1}{2}(n+1)n$$

# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

$$\sum_{i=1..n} i =$$

$$1 + \ldots + \underbrace{n/2 + \ldots + n}$$

Note that the final term is within a constant multiple of the middle term:

$$\frac{f(n)}{f(n/2)} = \frac{n}{n/2} = 2$$

Thus half the terms are roughly the same and the sum is roughly the number of terms, n, times this value

$$\sum_{i=1..n} i = \theta(n \cdot n)$$

# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

Is the statement true for this function?

$$\sum_{i=1..n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

# Arithmetic Like:

$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$

$\sum_{i=1..n} i =$

$1^2 + \ldots + (n/2)^2 + \ldots + n^2$

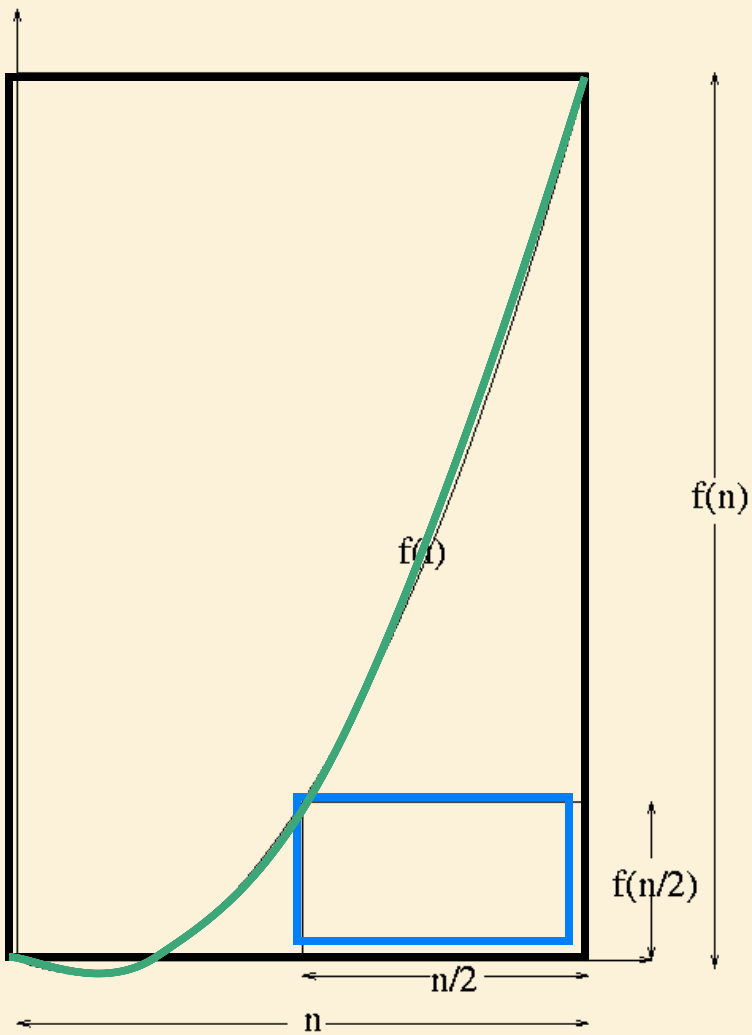
$^1/_4\, n^2$

Again half the terms are roughly the same.

$$\sum_{i=1..n} i^2 = \theta(n \cdot n^2)$$

# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$




area of small square $= {}^{n}/_{2} \cdot f({}^{n}/_{2})$

$\le \sum_{i=1..n} f(i)$

$\approx$ area under curve

$\le$ area of big square $= n \cdot f(n)$

# Arithmetic Like:

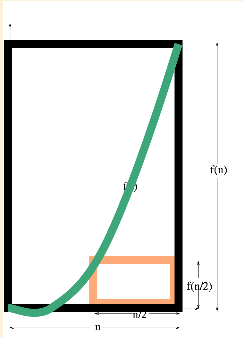
$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

$f(n) = n^2$

$$\sum_{i=1..n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

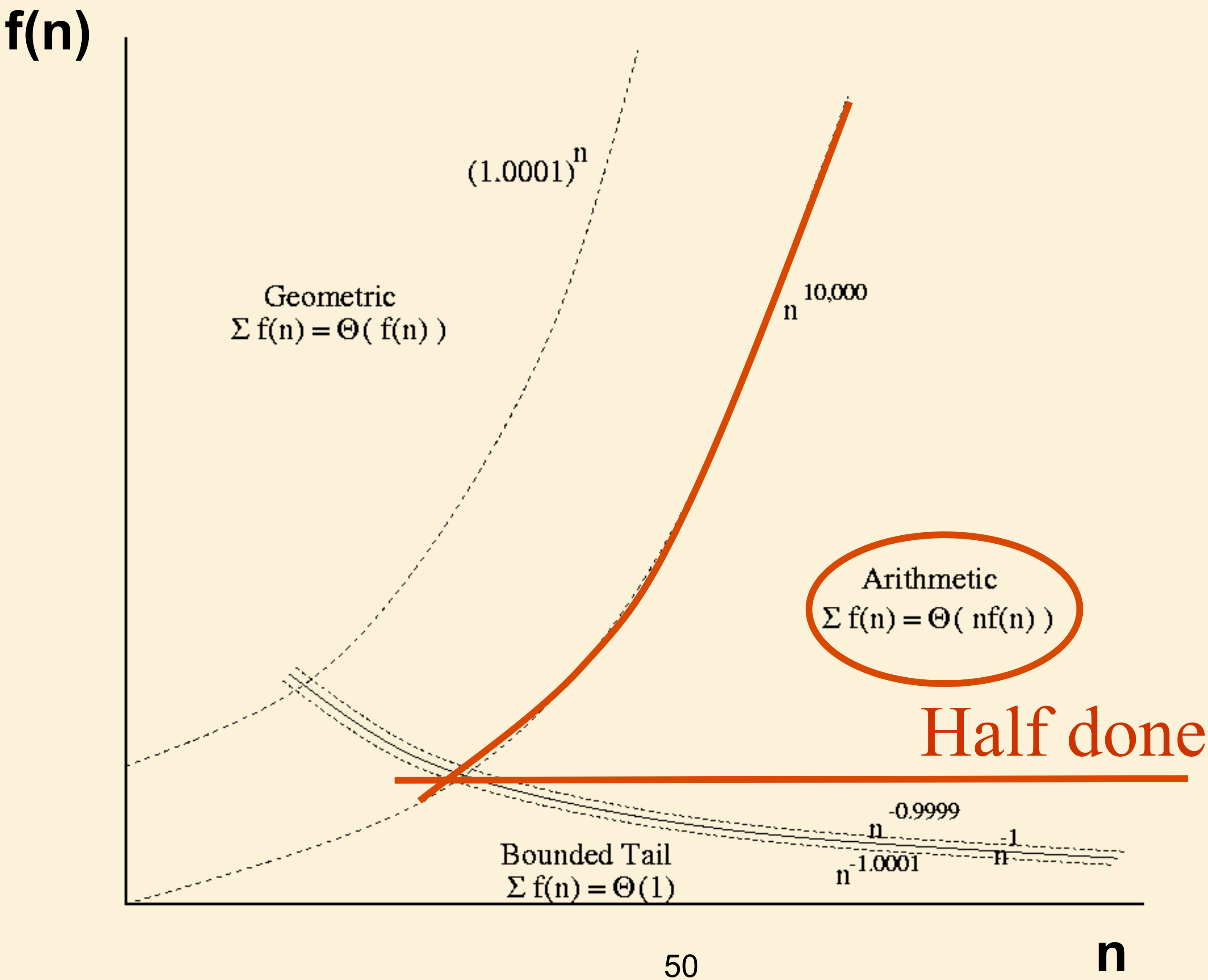
$= ?$

The key property is

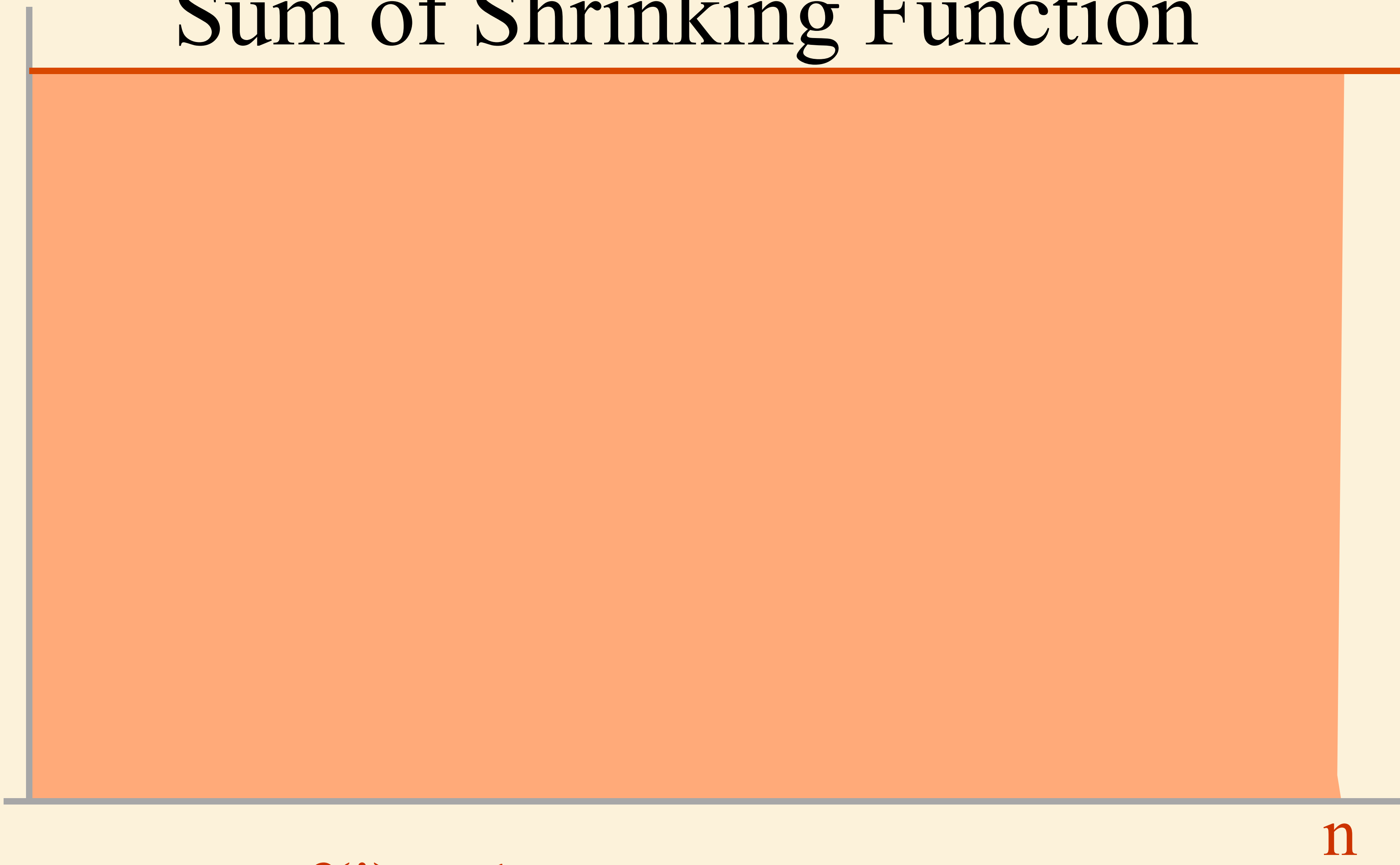



$$f(n/_2) = \theta(f(n))$$

# Adding Made Easy



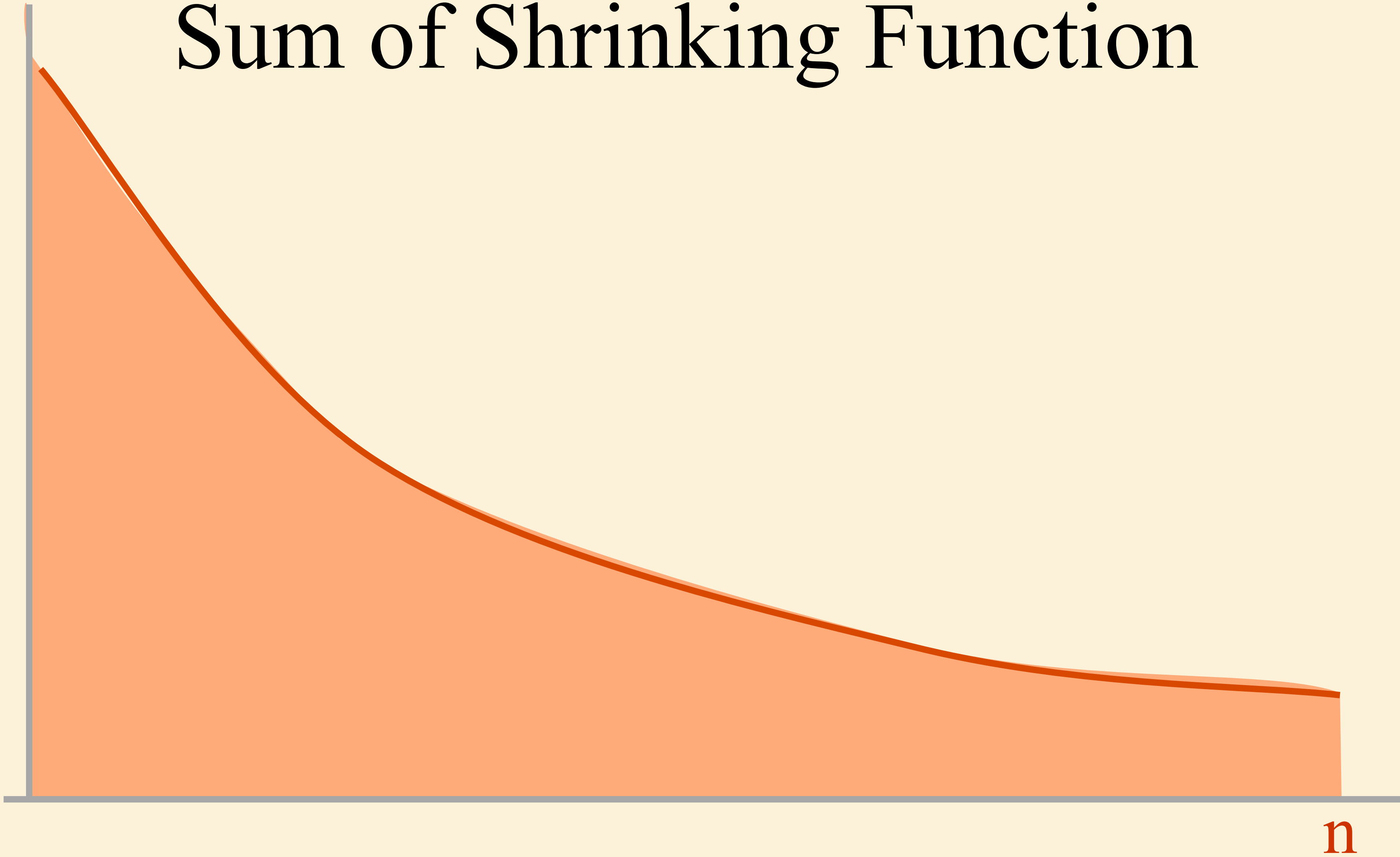
# Sum of Shrinking Function




$f(i) = 1$

$\sum_{i=1..n} f(i) = \theta(n)$

# Sum of Shrinking Function



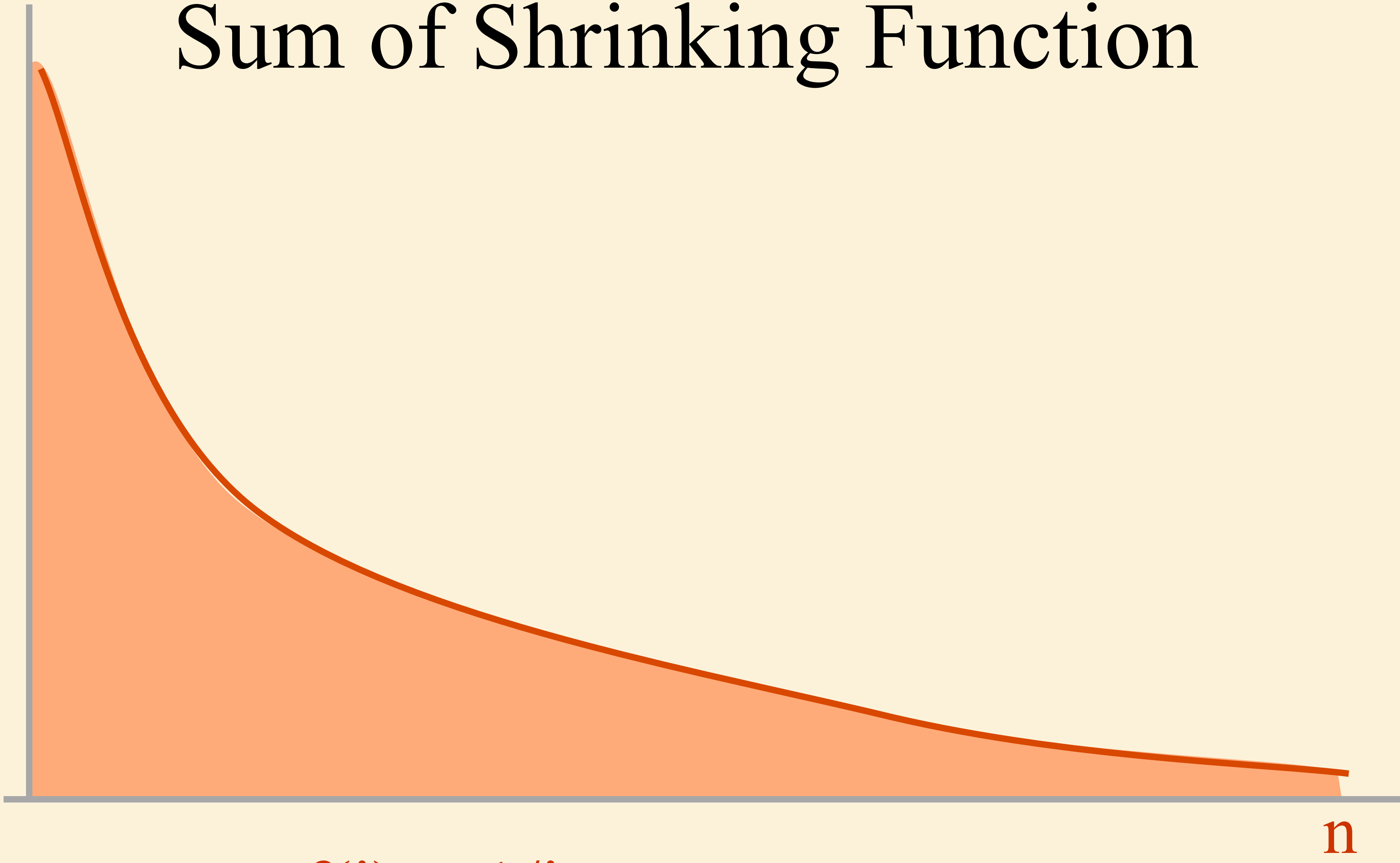

$f(i) = 1/i^{1/2}$

$\sum_{i=1..n} f(i) = \theta(n^{1/2})$

$f(n) \in n^{\Theta(1)-1}$

$\frac{1}{2} \in \Theta(1)$

# Sum of Shrinking Function




$$f(i) = 1/i$$

$$\sum_{i=1..n} f(i) = \theta(\log n)$$

# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

Does the statement hold
for the Harmonic Sum?

$$\sum_{i=1..n} {}^{1}/_{i} = {}^{1}/_{1} + {}^{1}/_{2} + {}^{1}/_{3} + {}^{1}/_{4} + {}^{1}/_{5} + \ldots + {}^{1}/_{n}$$

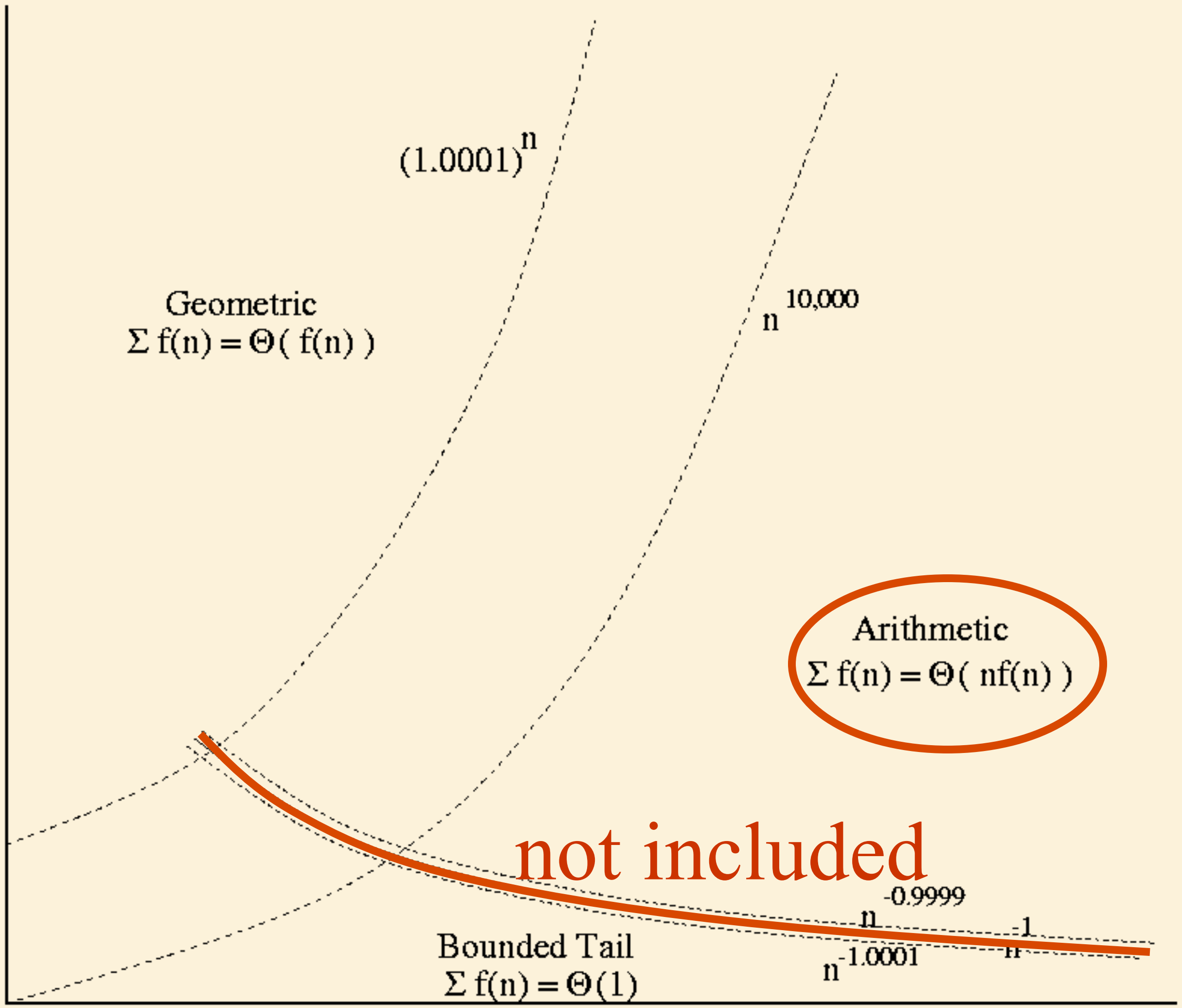
$$\sum_{i=1..n} f(i) = \sum_{i=1..n} {}^{1}/_{i} = \theta(\log n)$$

$$\neq$$

$$\theta(1) = \theta(n \cdot {}^{1}/_{n}) = \theta(n \cdot f(n))$$

No the statement does not hold! $\left(\frac{1}{n} \notin n^{\Theta(1)-1}\right)$

# Adding Made Easy



# Arithmetic Like:

$$f(n) = n^{\theta(1)-1} \Rightarrow \sum_{i=1..n} f(i) = \theta(n \cdot f(n))$$

Upper Extreme: $\sum_{i=1..n} i^{1000} = {}^{1}/_{1001}\, n^{1001}$

$= {}^{1}/_{1001}\, n \cdot f(n)$

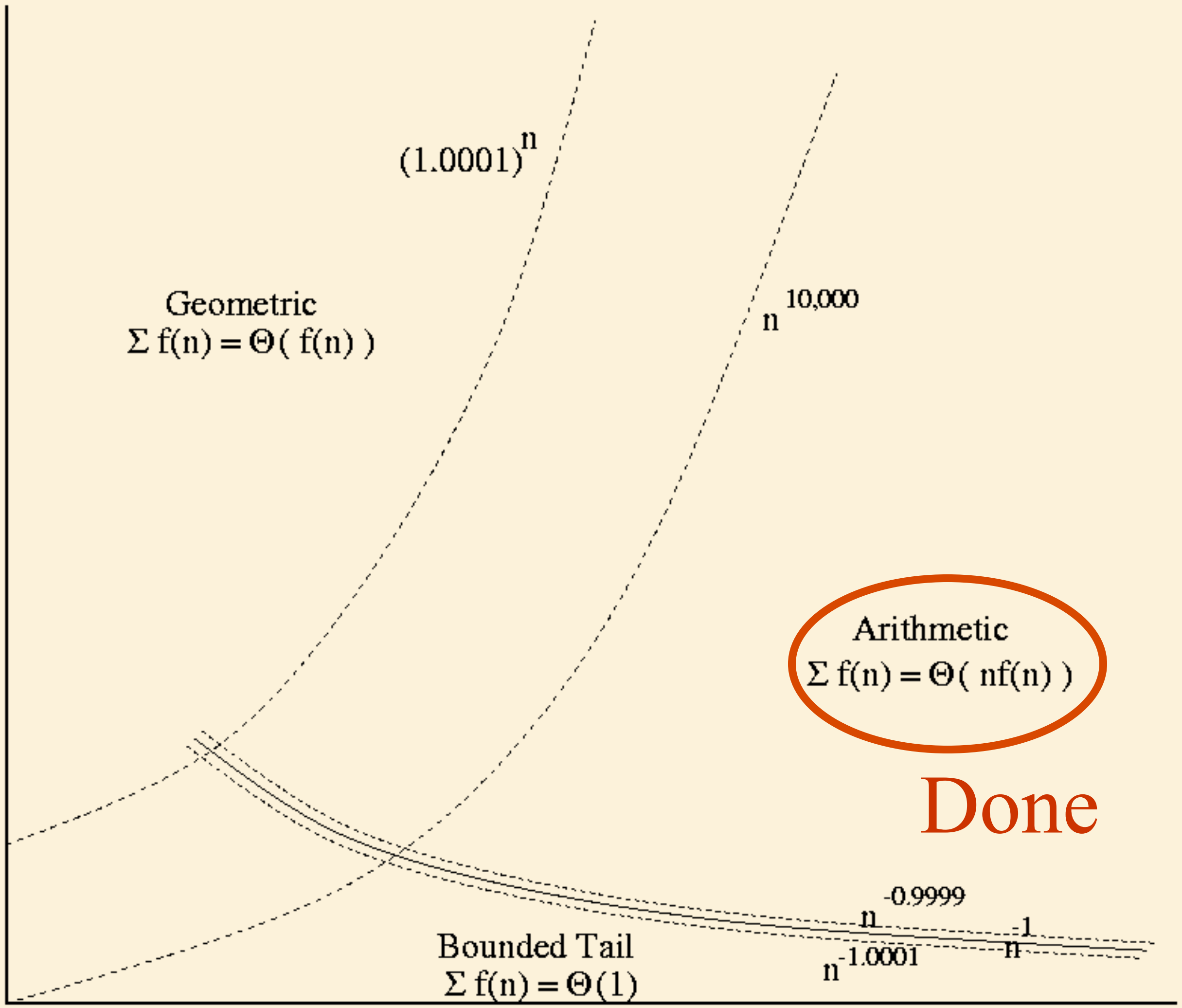
Intermediate Case: $\sum_{i=1..n} 1 = n \cdot 1$

$= n \cdot f(n)$

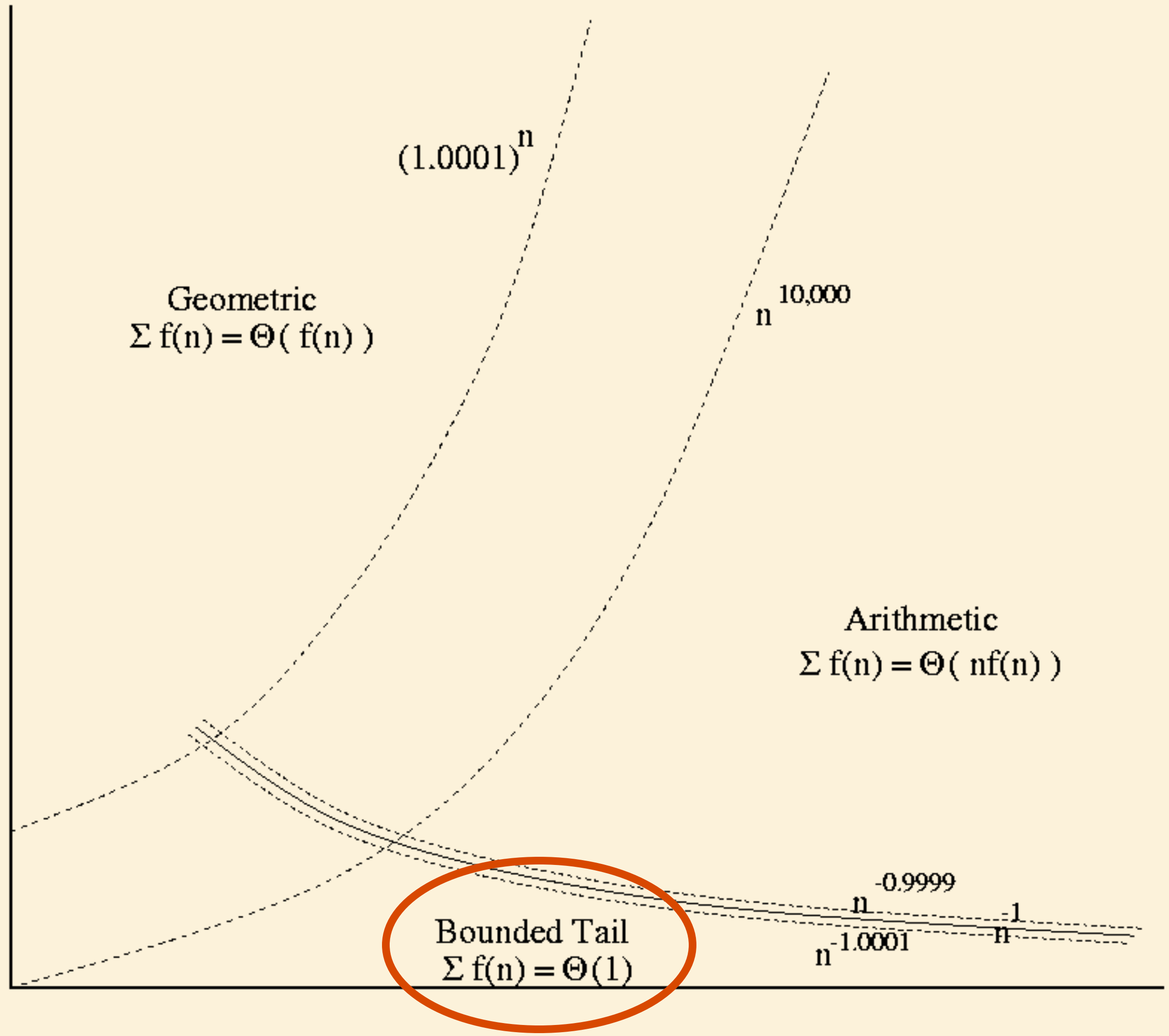
Lower Extreme: $\sum_{i=1..n} {}^{1}/_{i^{0.9999}} = \theta(n^{0.0001})$

$= \theta(n \cdot f(n))$

# Adding Made Easy



# Adding Made Easy



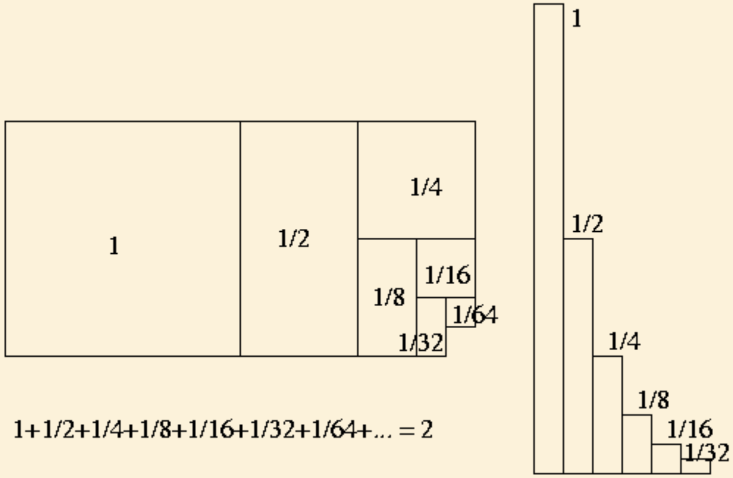
# Bounded Tail:

$f(n) \le n^{-1-\Omega(1)} \Rightarrow \sum_{i=1..n} f(i) = \theta(1)$

If the terms f(i) decrease towards zero sufficiently quickly,
then the sum will be a constant.

The classic example

$\sum_{i=0..n} {}^{1}/_{2^i} = 1 + {}^{1}/_{2} + {}^{1}/_{4} + {}^{1}/_{8} + \ldots < 2.$



# Bounded Tail:

$f(n) \leq n^{-1-\Omega(1)} \Rightarrow \sum_{i=1..n} f(i) = \theta(1)$

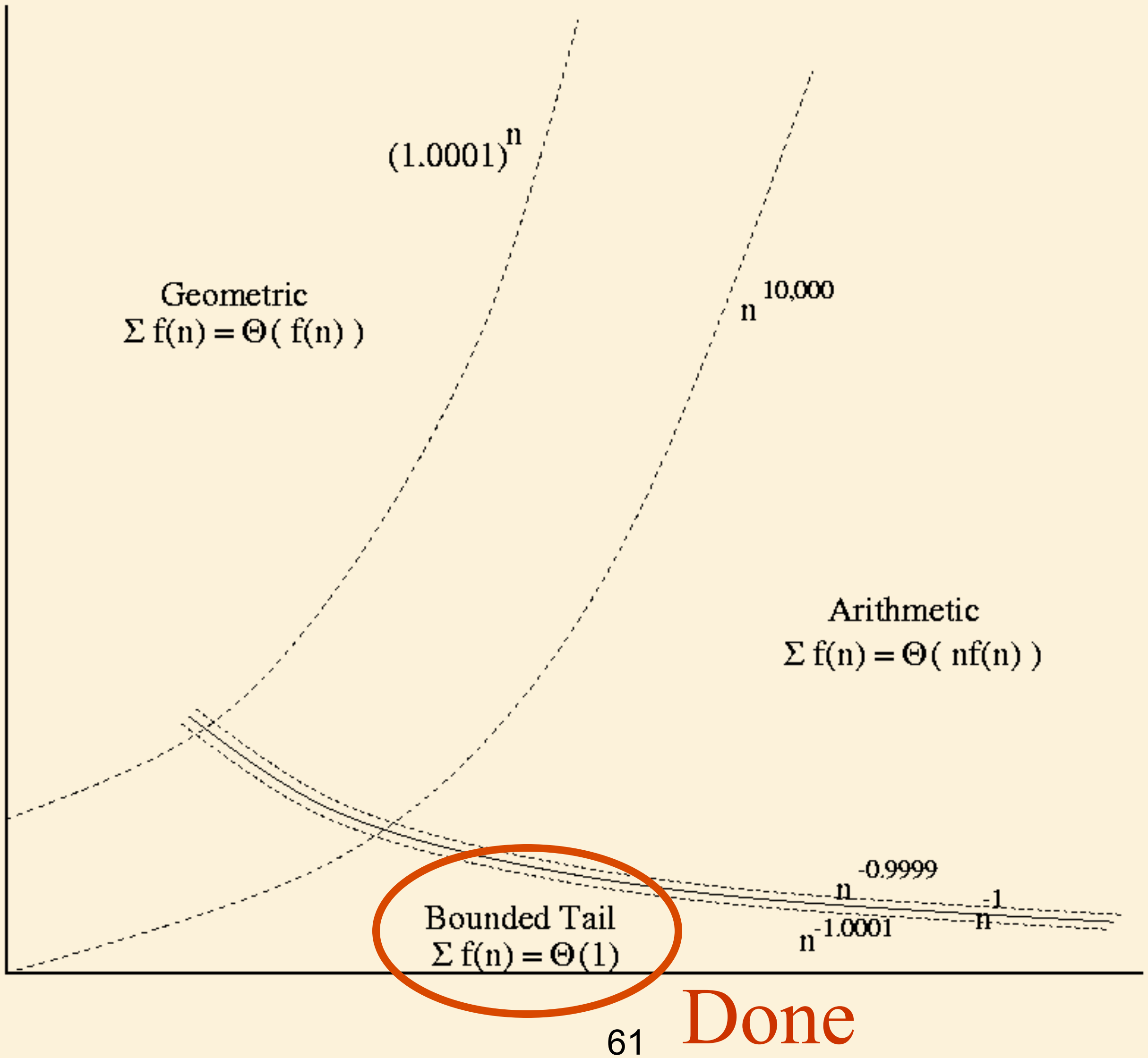
Upper Extreme: $\sum_{i=1..n} {}^{1}/_{i^{1.0001}} = \theta(1)$

All functions in between.

- $\sum_{i=1}^{n} \frac{1}{i^{1+\epsilon}} \approx \frac{1}{\epsilon} = \Theta(1)$, for all constants $\epsilon > 0$.
- $\sum_{i=1}^{n} \frac{1}{i^2} = \Theta(1)$.
- $\sum_{i=1}^{n} \frac{\log^3 i}{i^{1.6}+3i} = \Theta(1)$.
- $\sum_{i=1}^{n} \frac{n^{100}}{2^i} = \Theta(1)$.
- $\sum_{i=1}^{n} \frac{1}{2^{2^i}} = \Theta(1)$.

No Lower Extreme: $\sum_{i=1..n} {}^{1}/_{2^{2^{2^i}}} = \theta(1).$

# Adding Made Easy



# Summary

- Geometric Like: If $f(n) \geq 2^{\Omega(n)}$, then $\sum_{i=1..n} f(i) = \theta(f(n))$.
- Arithmetic Like: If $f(n) = n^{\theta(1)-1}$, then $\sum_{i=1..n} f(i) = \theta(n \cdot f(n))$.
- Harmonic: If $f(n) = {}^{1}/_{n}$ , then $\sum_{i=1..n} f(i) = \theta(\log n)$.
- Bounded Tail: If $f(n) \leq n^{-1-\Omega(1)}$, then $\sum_{i=1..n} f(i) = \theta(1)$.

(For +, -, ·, ÷ , exp, log functions f(n))