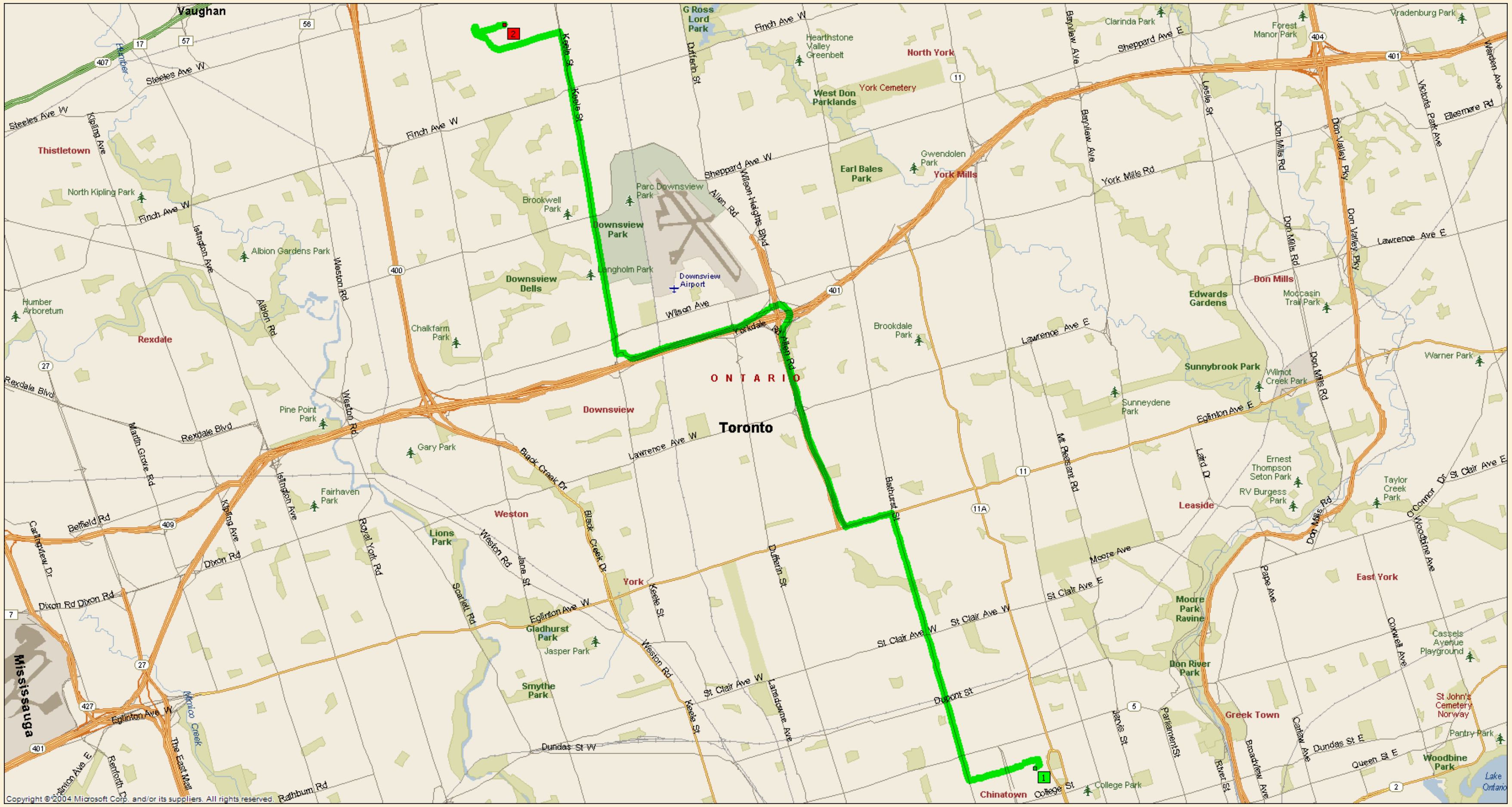
Graph Search Algorithms


Toronto
ONTARIO
Keele St
Allen Rd
Bathurst St
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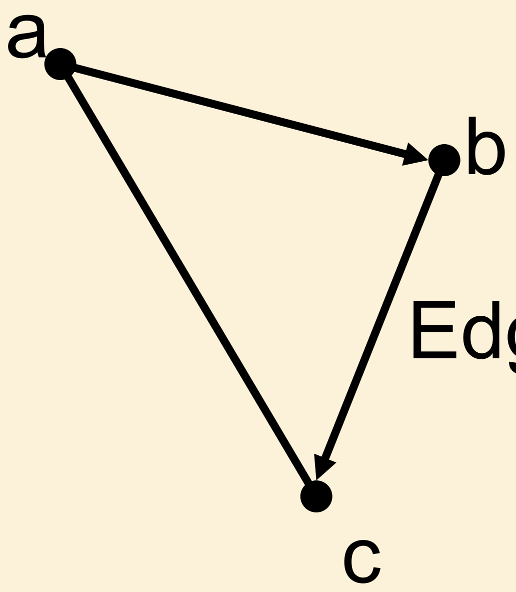
Graph



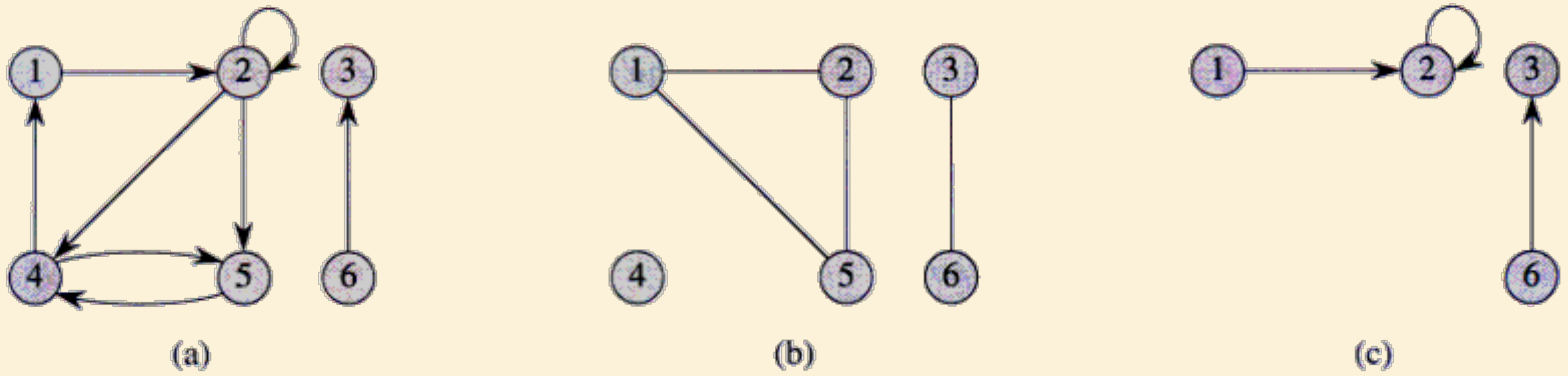
Node ~ city or computer

Edge ~ road or data cable

Undirected or Directed

A surprisingly large number of computational problems can be expressed as graph problems.

Directed and Undirected Graphs

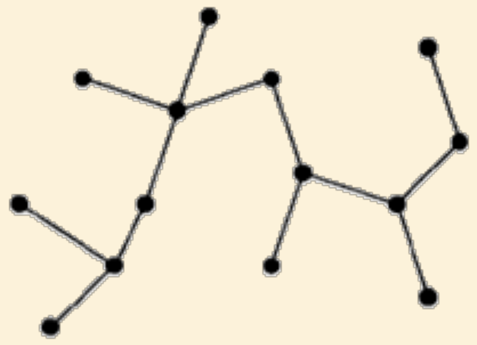


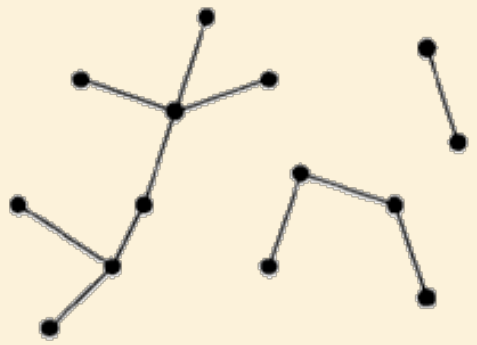
(a) A directed graph $G = (V, E)$, where $V = \{1,2,3,4,5,6\}$ and $E = \{(1,2), (2,2), (2,4), (2,5), (4,1), (4,5), (5,4), (6,3)\}$. The edge (2,2) is a self-loop.

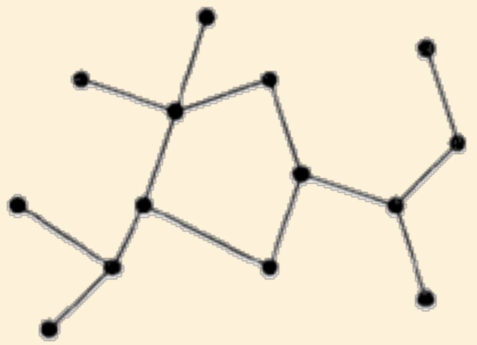
(b) An undirected graph $G = (V,E)$, where $V = \{1,2,3,4,5,6\}$ and $E = \{(1,2), (1,5), (2,5), (3,6)\}$. The vertex 4 is isolated.

(c) The subgraph of the graph in part (a) induced by the vertex set {1,2,3,6}.

Trees


Tree


Forest


Graph with Cycle

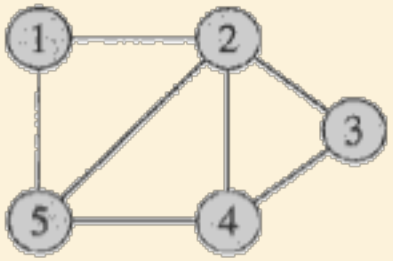
A tree is a connected, acyclic, undirected graph.

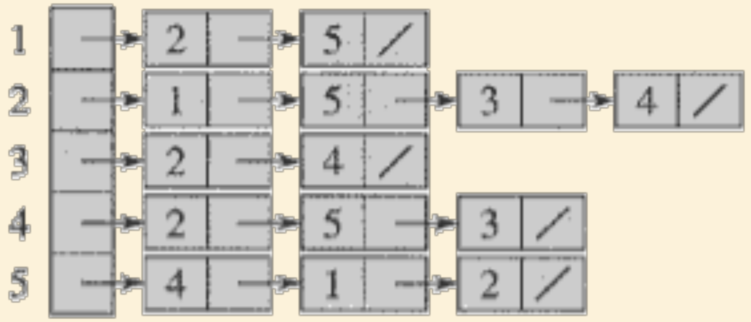
A forest is a set of trees (not necessarily connected)

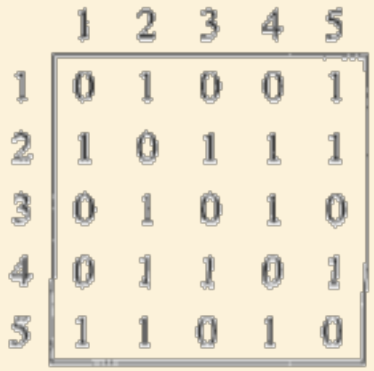
Running Time of Graph Algorithms

- Running time often a function of both |V| and |E|.
- For convenience, drop the | . | in asymptotic notation, e.g. $O(V+E)$.

Representations: Undirected Graphs



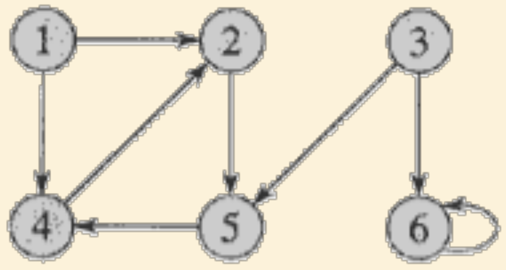


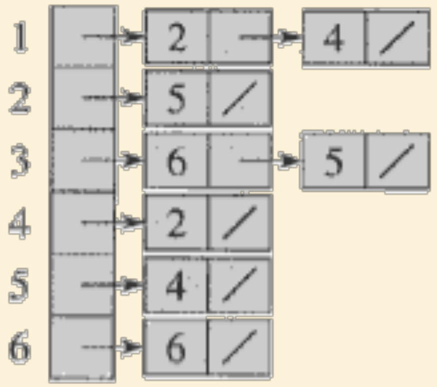


$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \end{array}$$

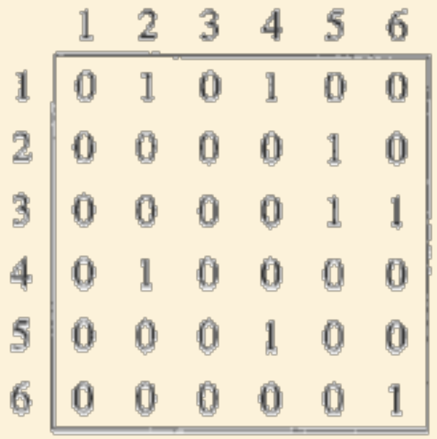
	Adjacency List	Adjacency Matrix
Space complexity:	$\theta(V+E)$	$\theta(V^2)$
Time to find all neighbours of vertex u:	$\theta(\text{degree}(u))$	$\theta(V)$
Time to determine if $(u,v)\in E$:	$\theta(\text{degree}(u))$	$\theta(1)$

Representations: Directed Graphs





Adjacency List



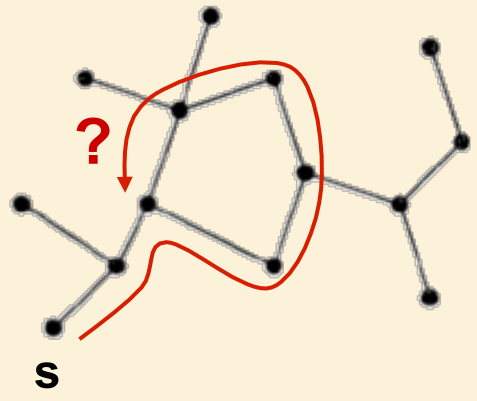
	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Adjacency Matrix

Space complexity:	$\theta(V+E)$	$\theta(V^2)$
Time to find all neighbours of vertex u:	$\theta(\text{degree}(u))$	$\theta(V)$
Time to determine if $(u,v)\in E$:	$\theta(\text{degree}(u))$	$\theta(1)$

Breadth-First Search

- Goal: To recover the shortest paths from a source node *s* to all other reachable nodes *v* in a graph.
  - The length of each path and the paths themselves are returned.
- Notes:
  - There are an exponential number of possible paths
  - This problem is harder for general graphs than trees because of cycles!



Breadth-First Search

Input: Graph $G = (V, E)$ (directed or undirected) and source vertex $s \in V$.

Output:

$d[v] =$ shortest path distance $\delta(s,v)$ from s to v, $\forall v \in V$.

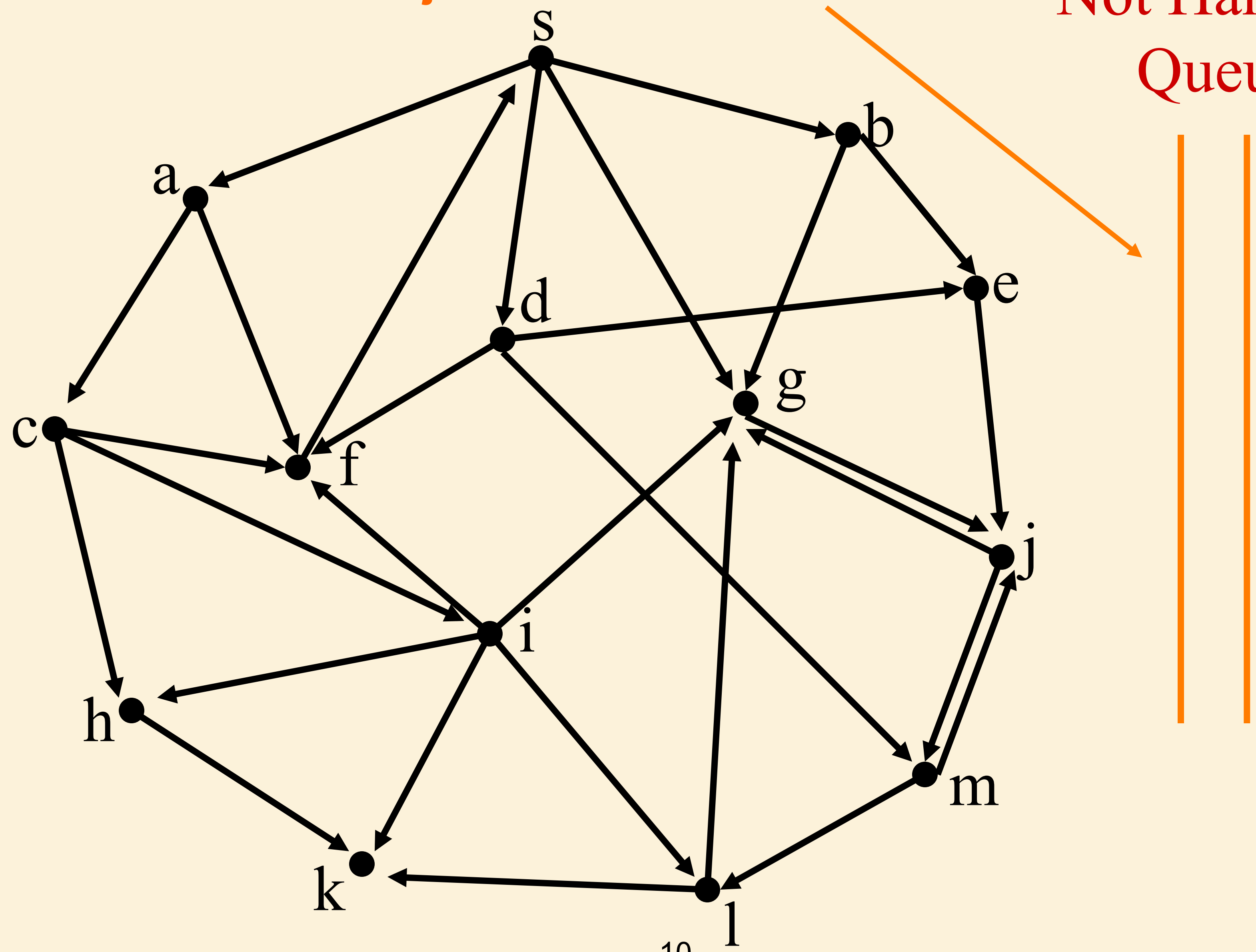
$\pi[v] = u$ such that (u,v) is last edge on a shortest path from s to v.

- Idea: send out search 'wave' from s.
- Keep track of progress by colouring vertices:
  - **Undiscovered** vertices are coloured **black**
  - **Just discovered** vertices (on the wavefront) are coloured **red.**
  - **Previously discovered** vertices (behind wavefront) are coloured **grey.**

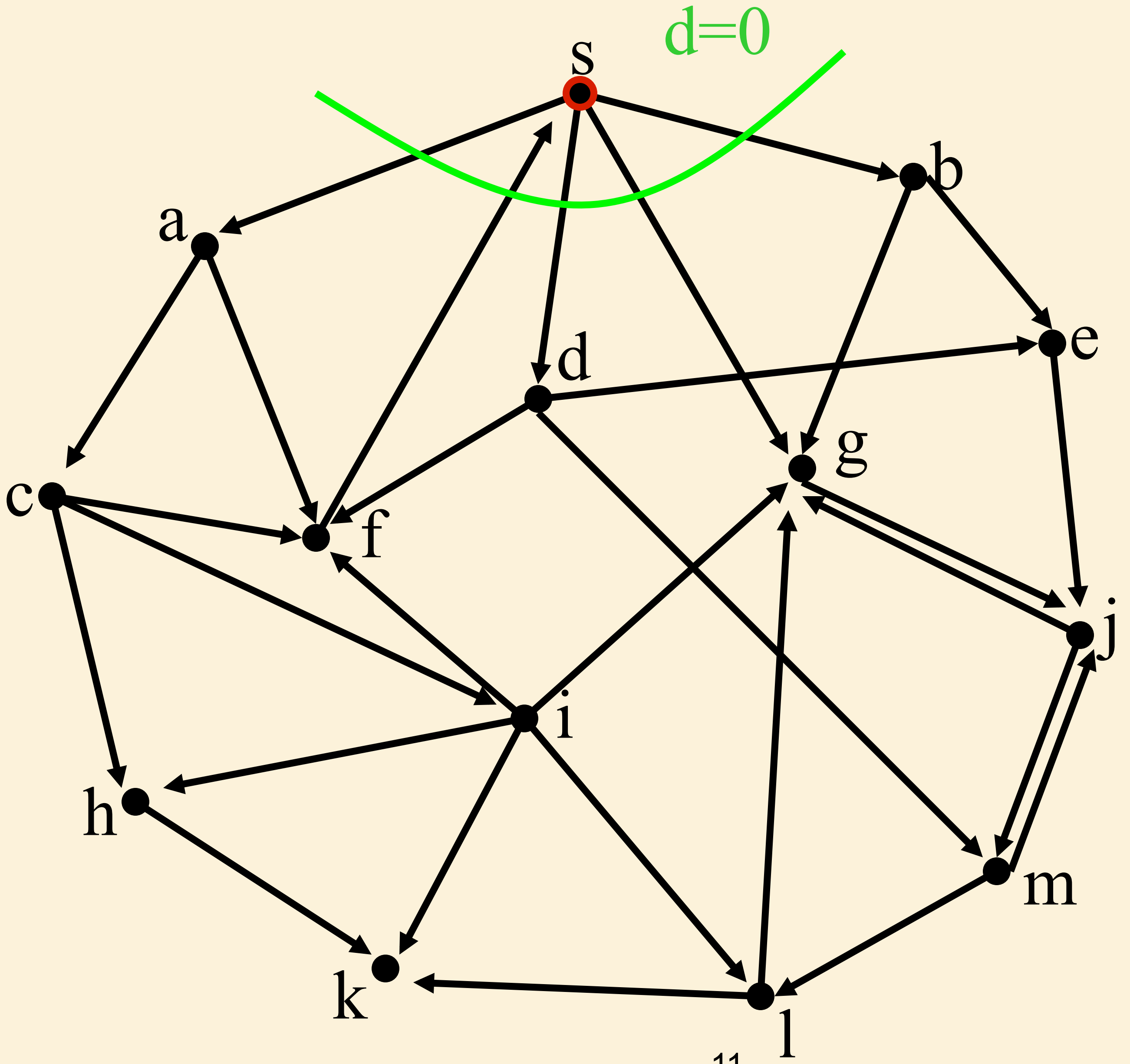
BFS

First-In First-Out (FIFO) queue stores 'just discovered' vertices

Found
Not Handled
Queue



BFS



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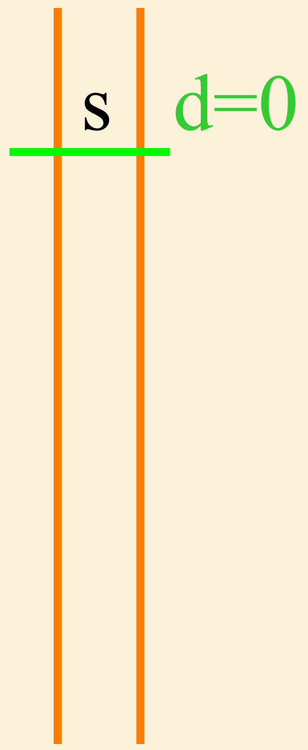
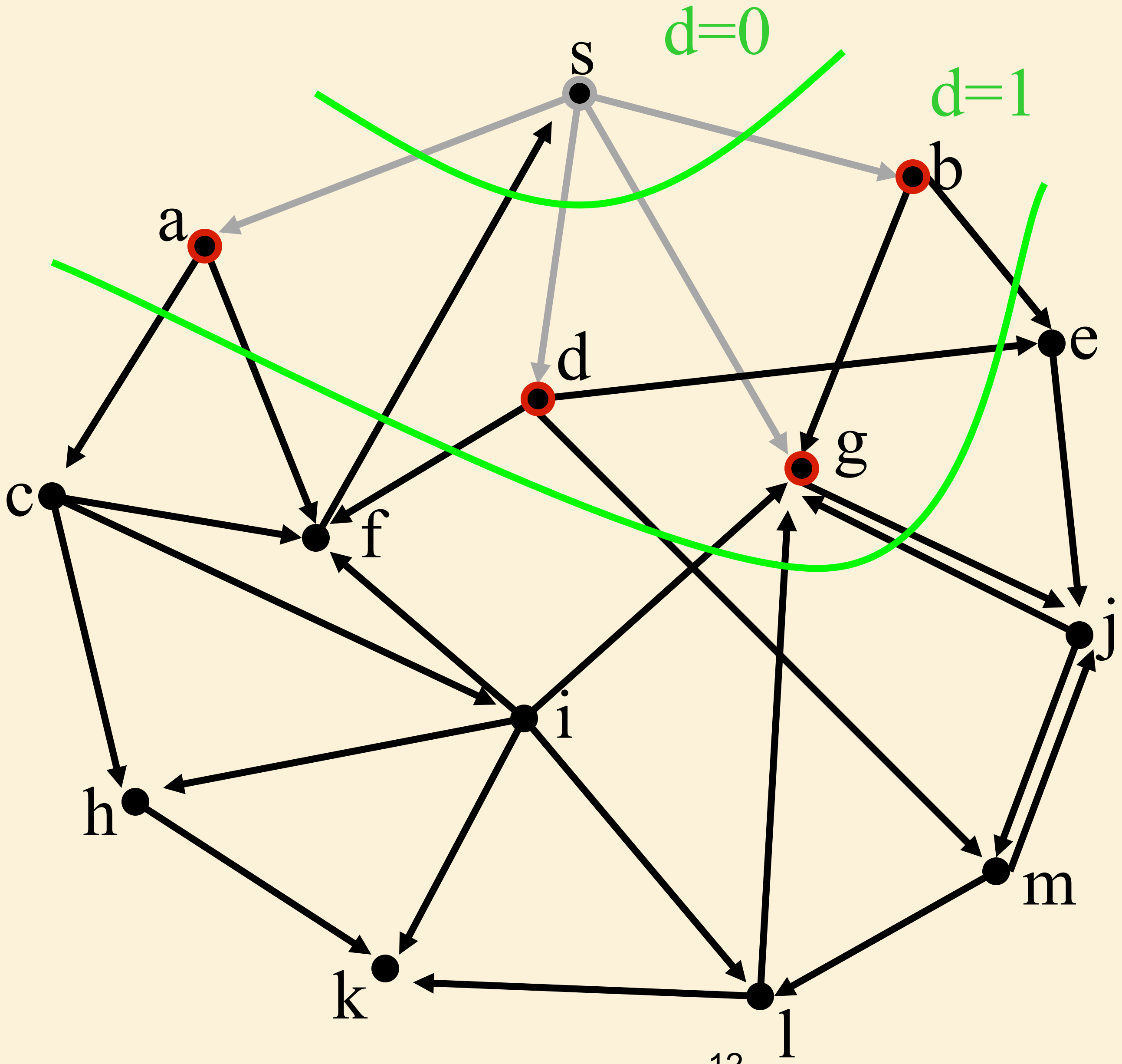


BFS



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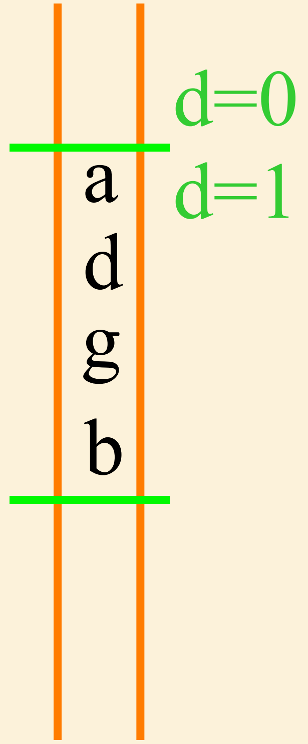

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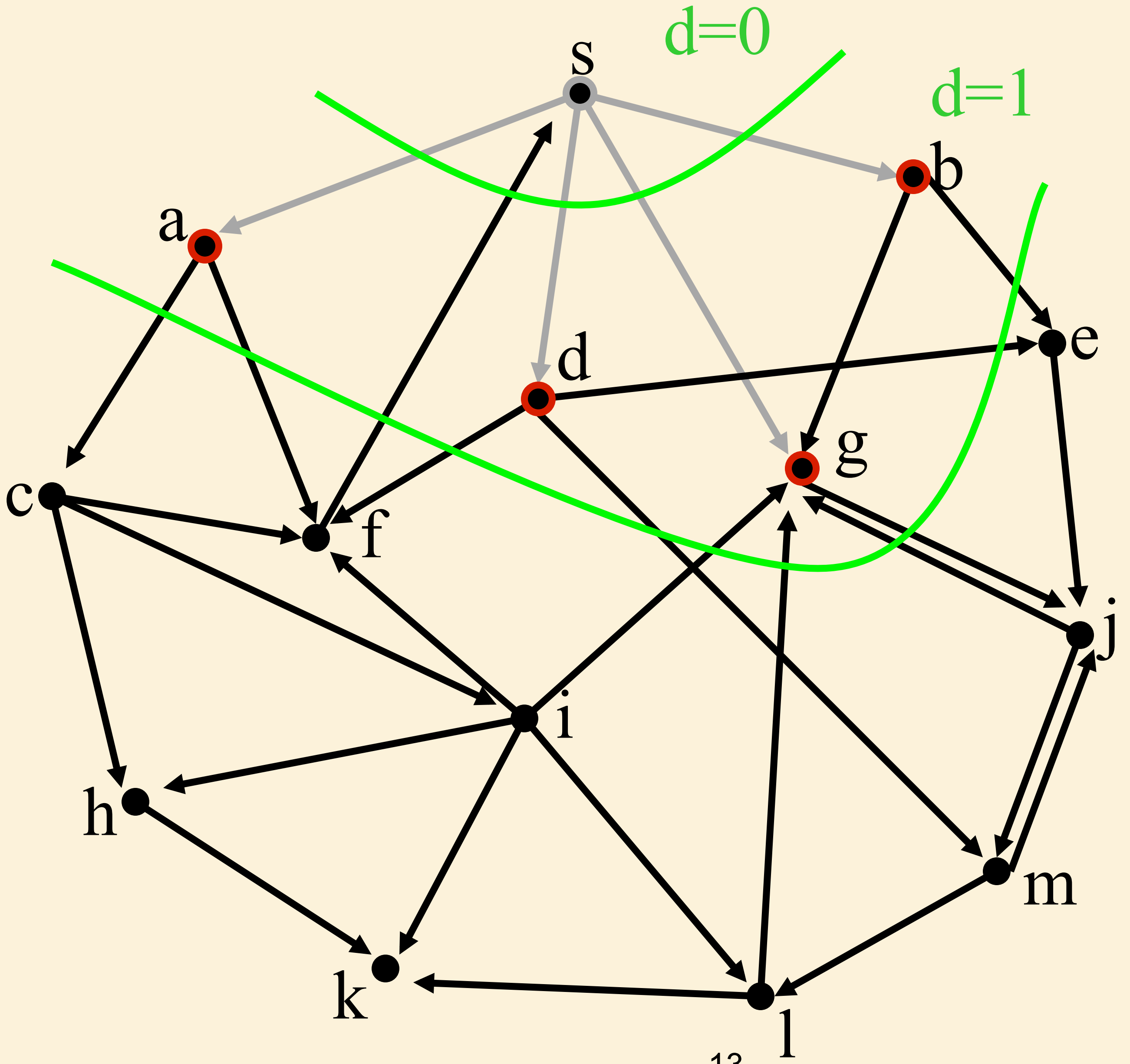

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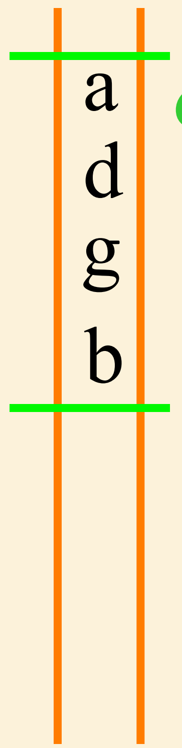

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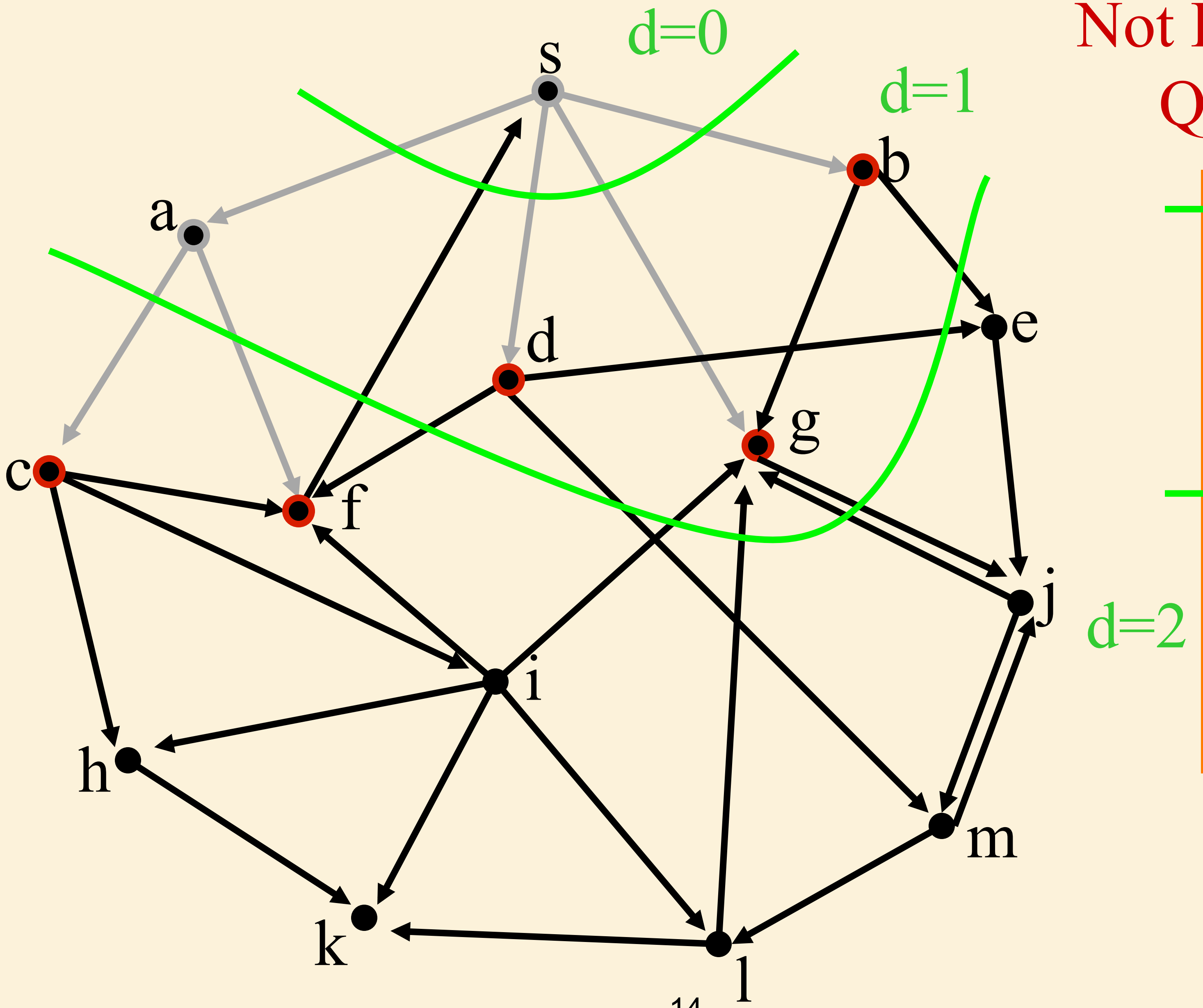

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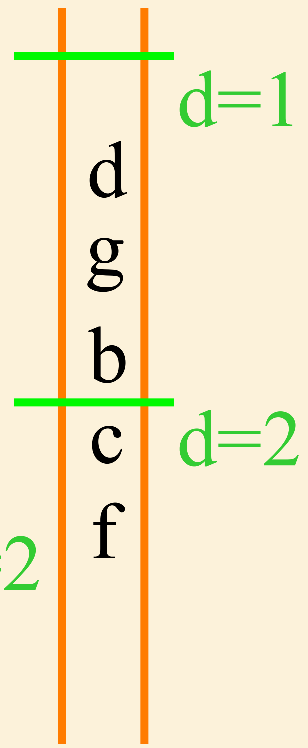

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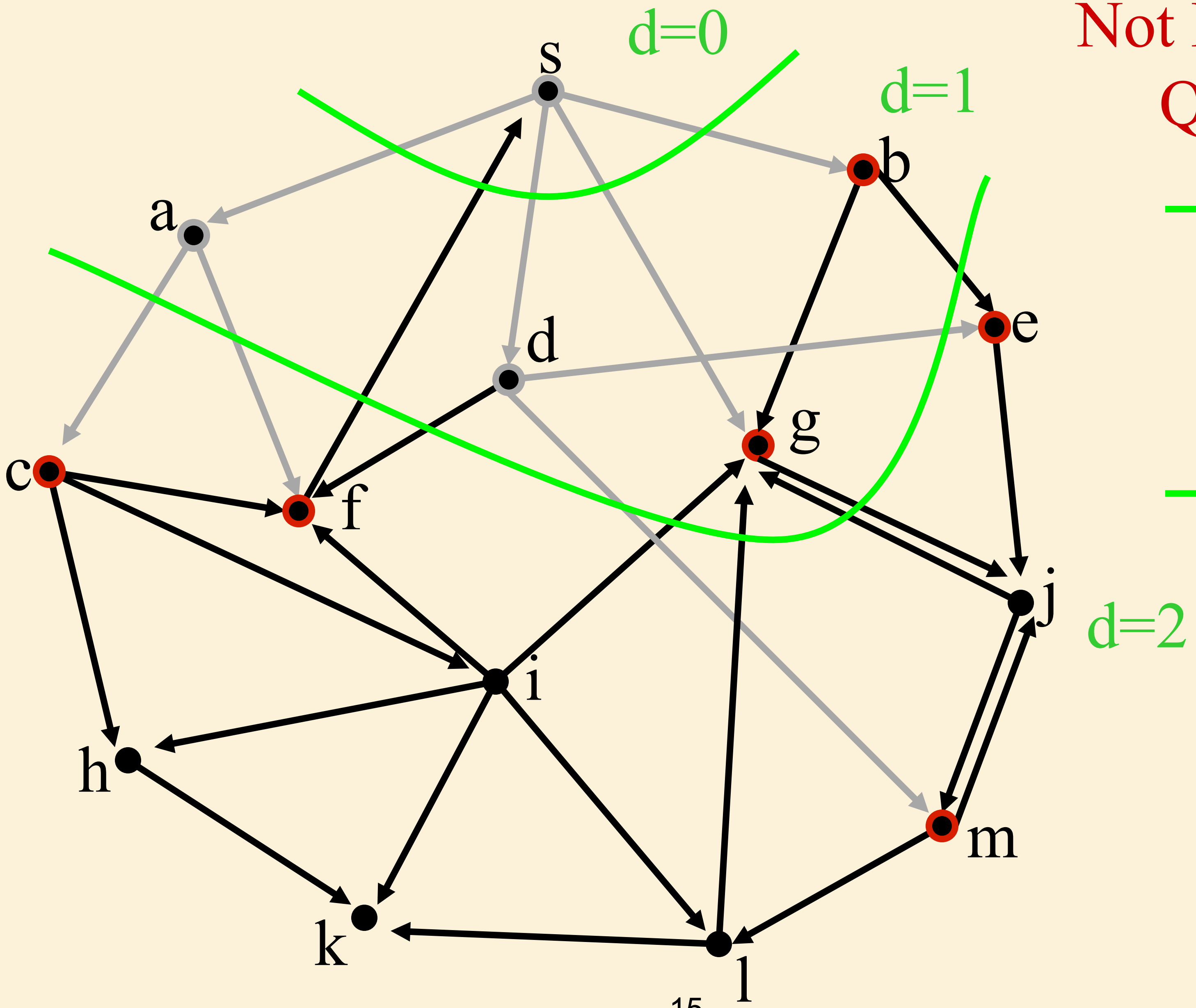

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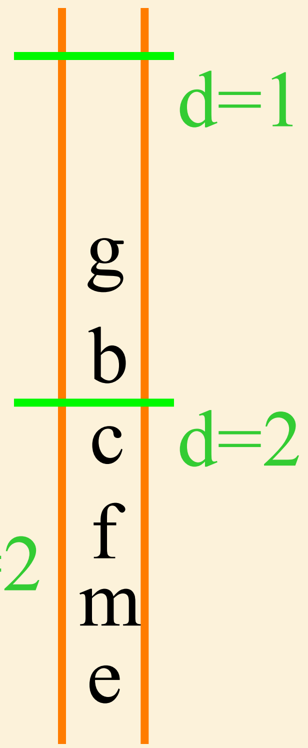

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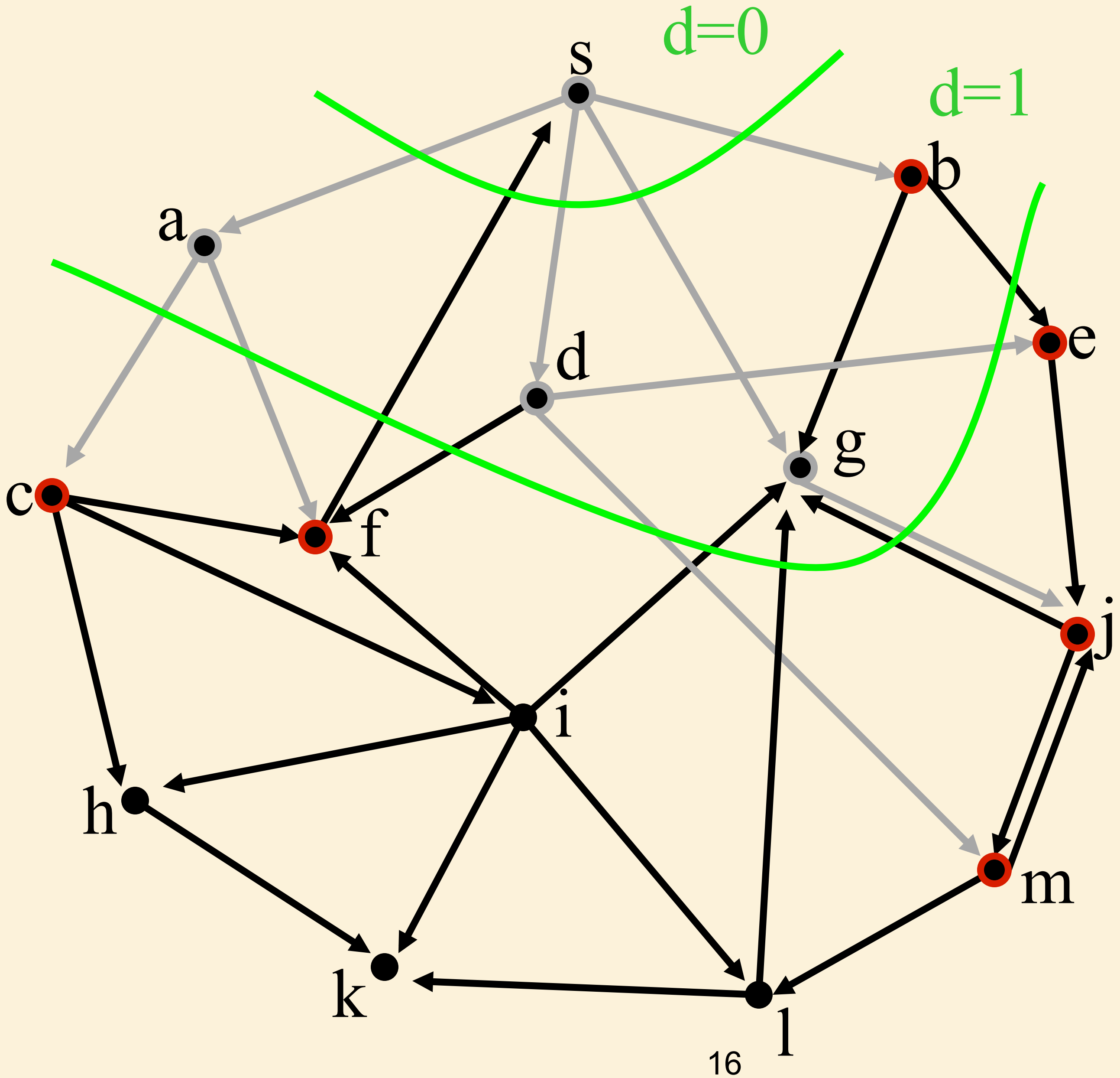

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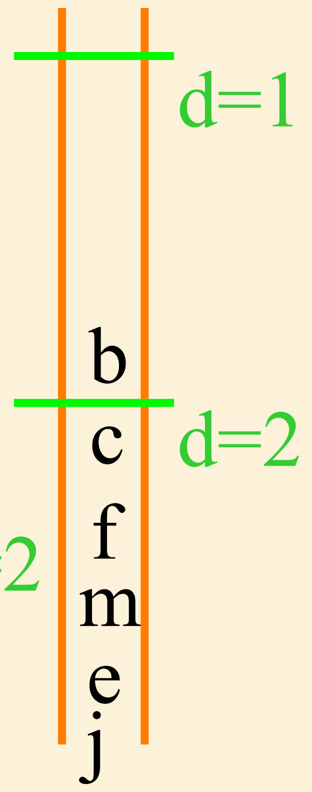

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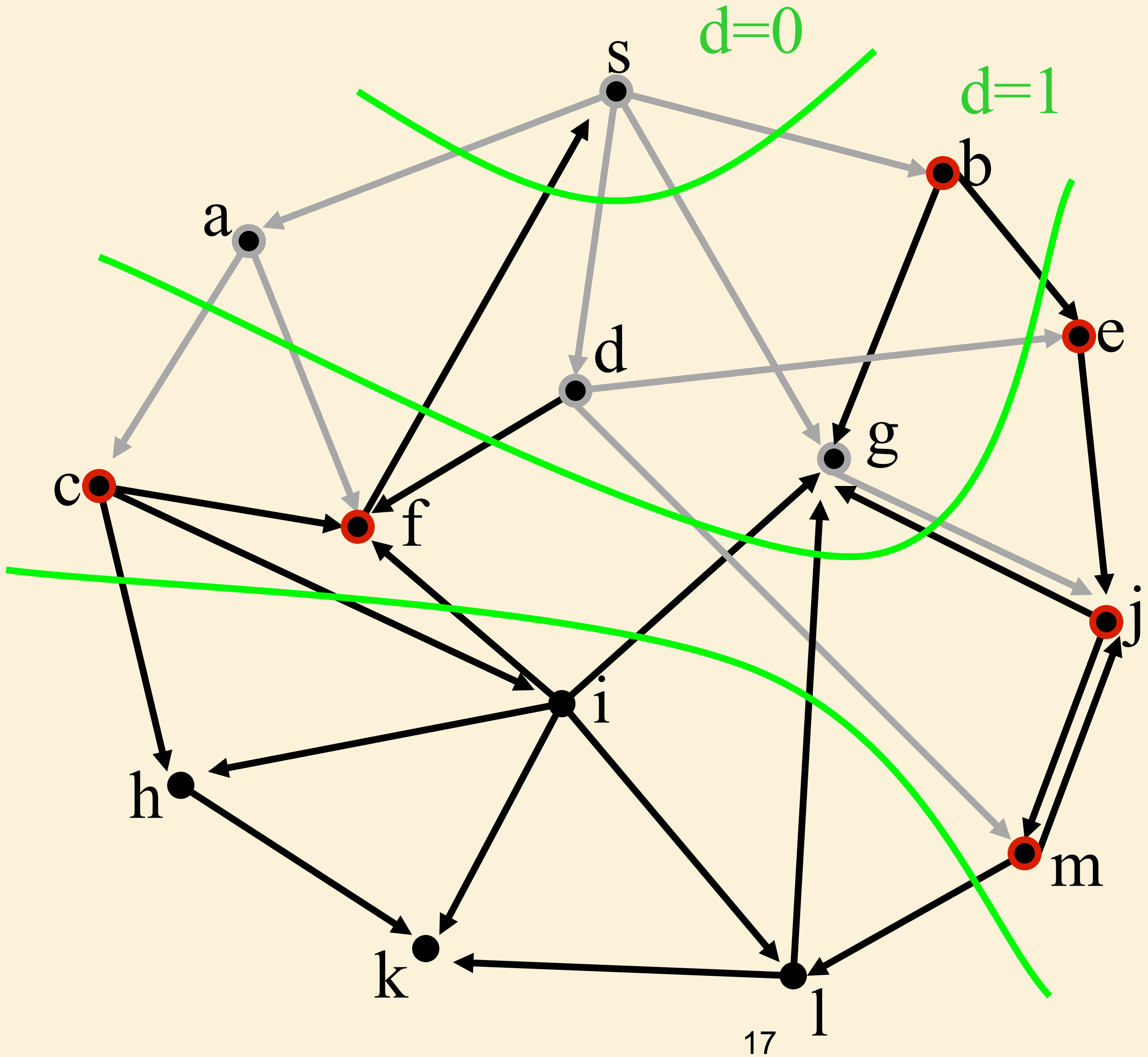

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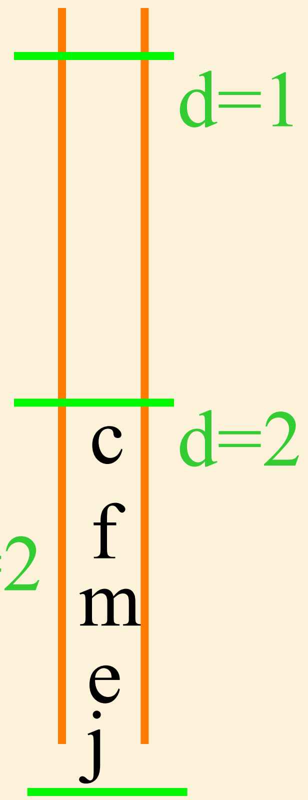

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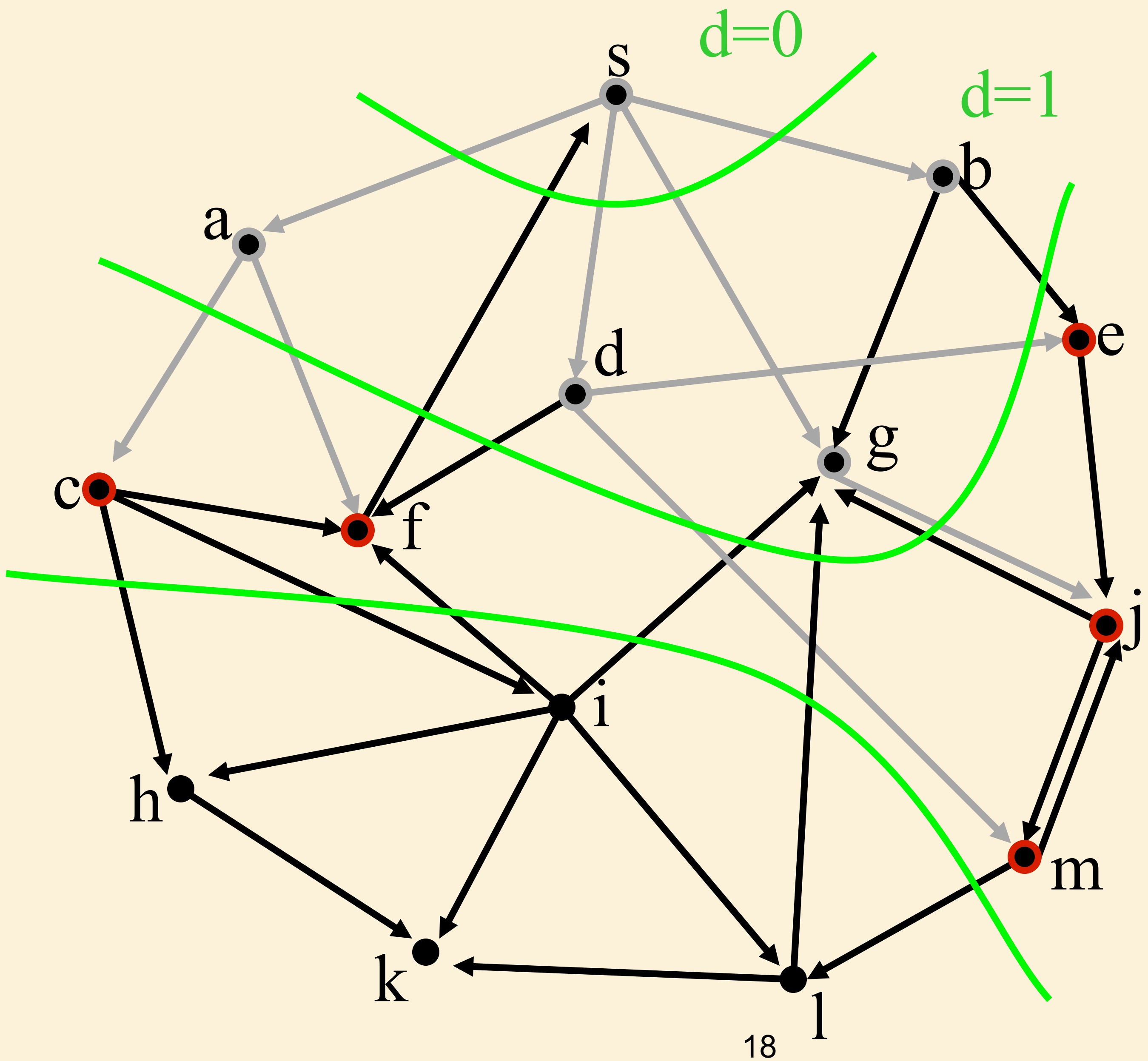

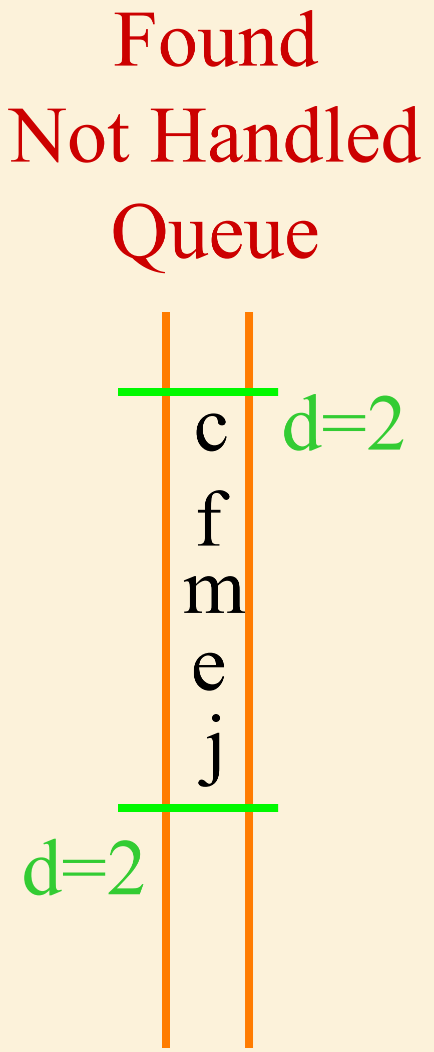

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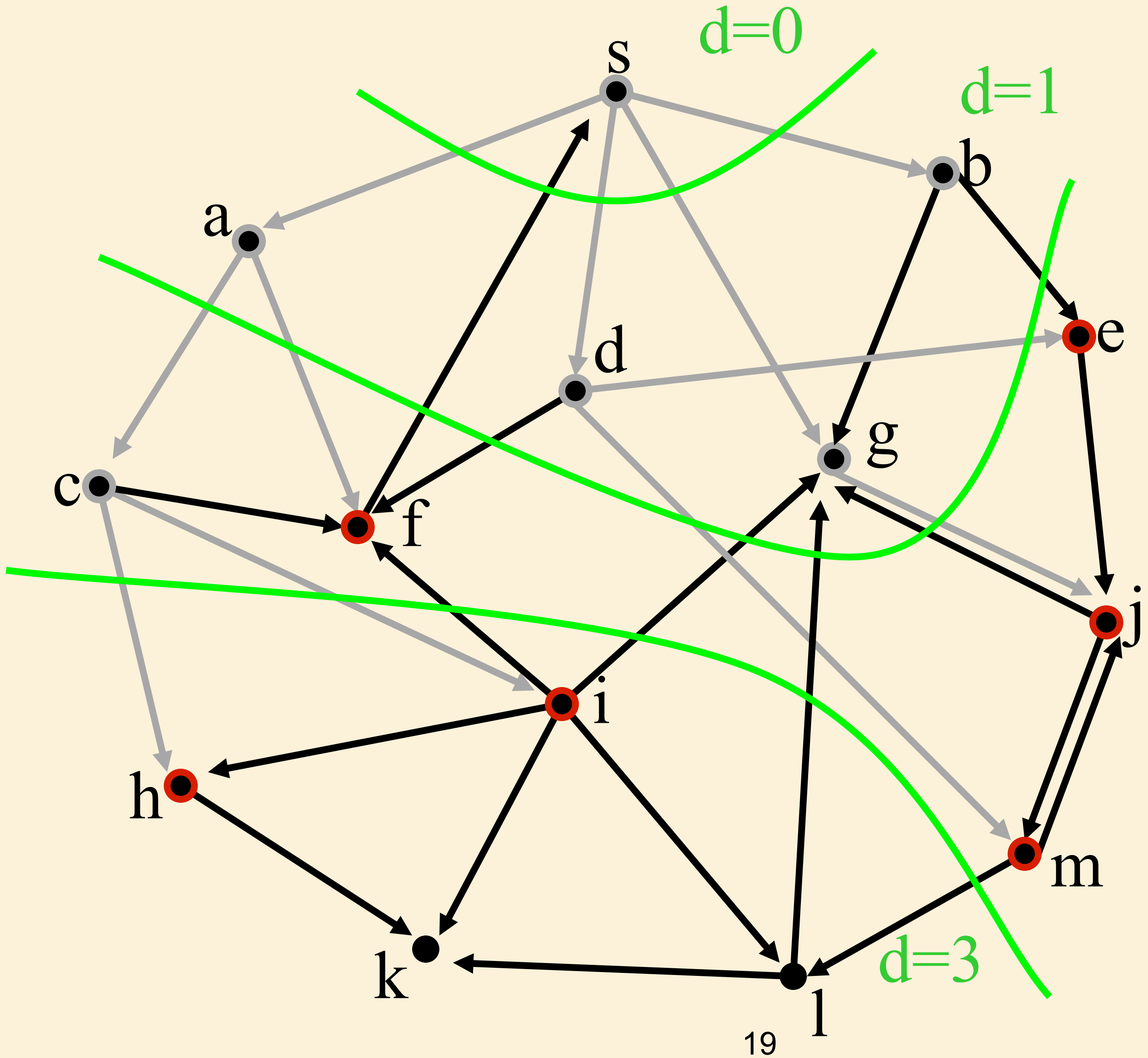

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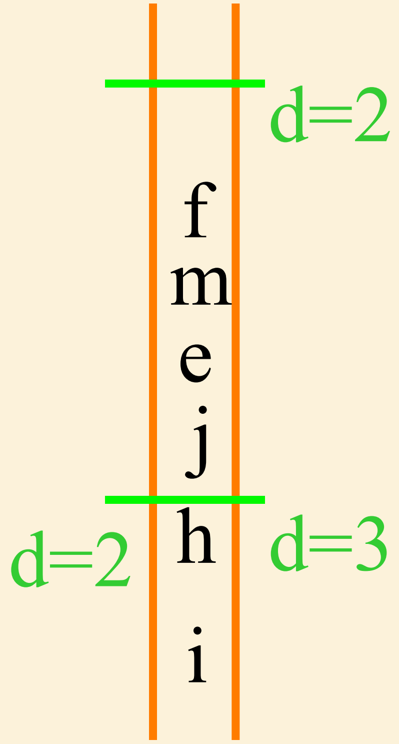

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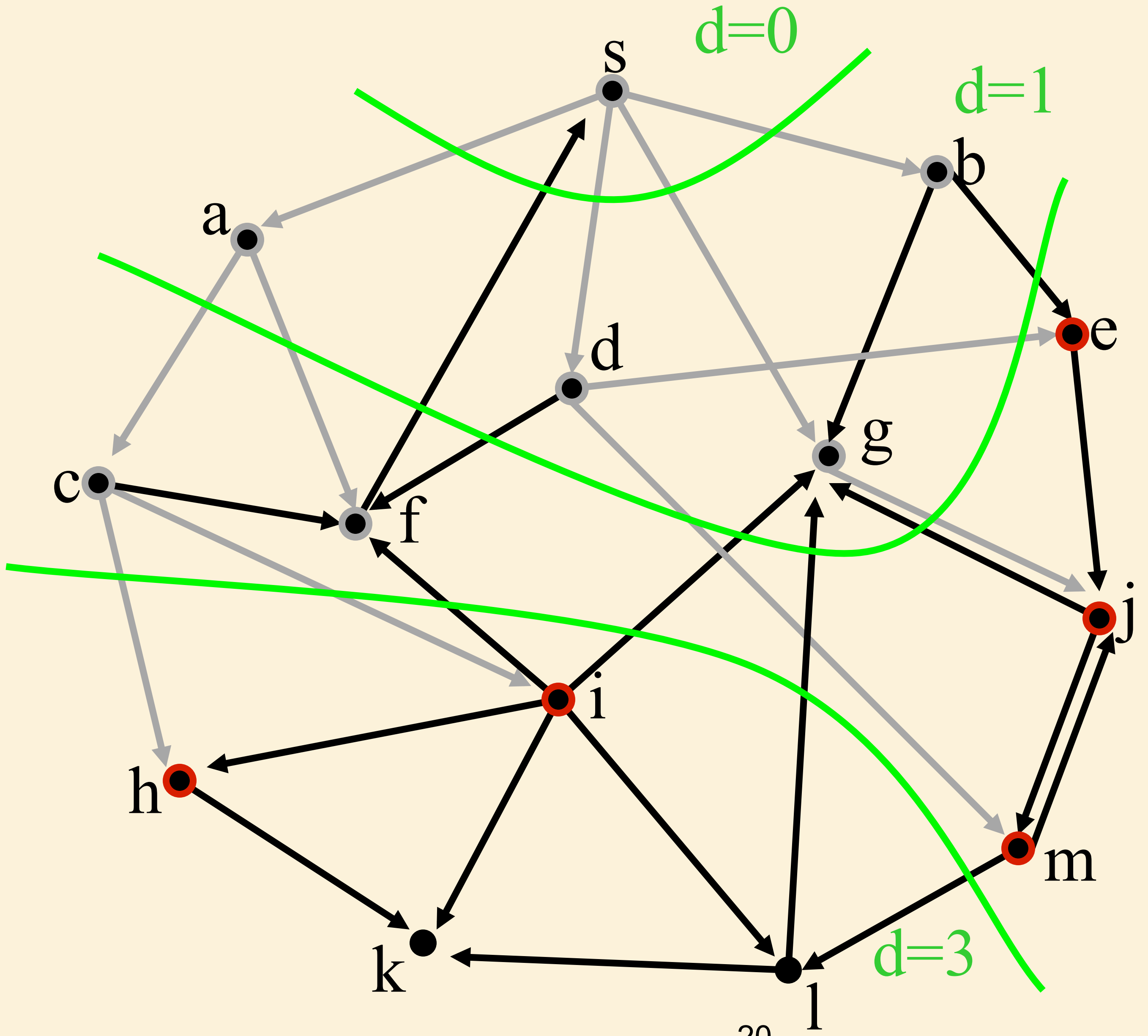

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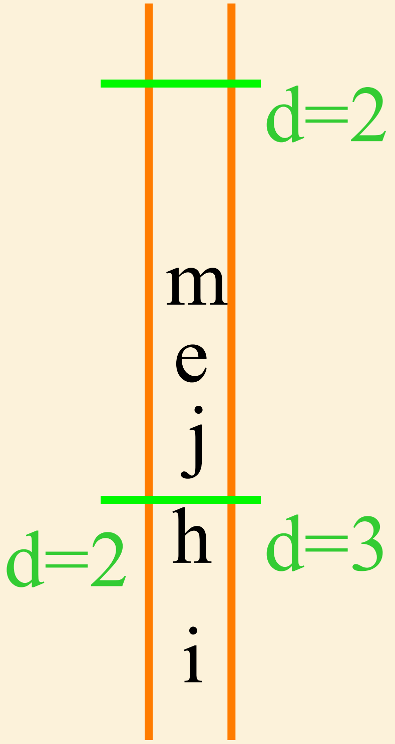

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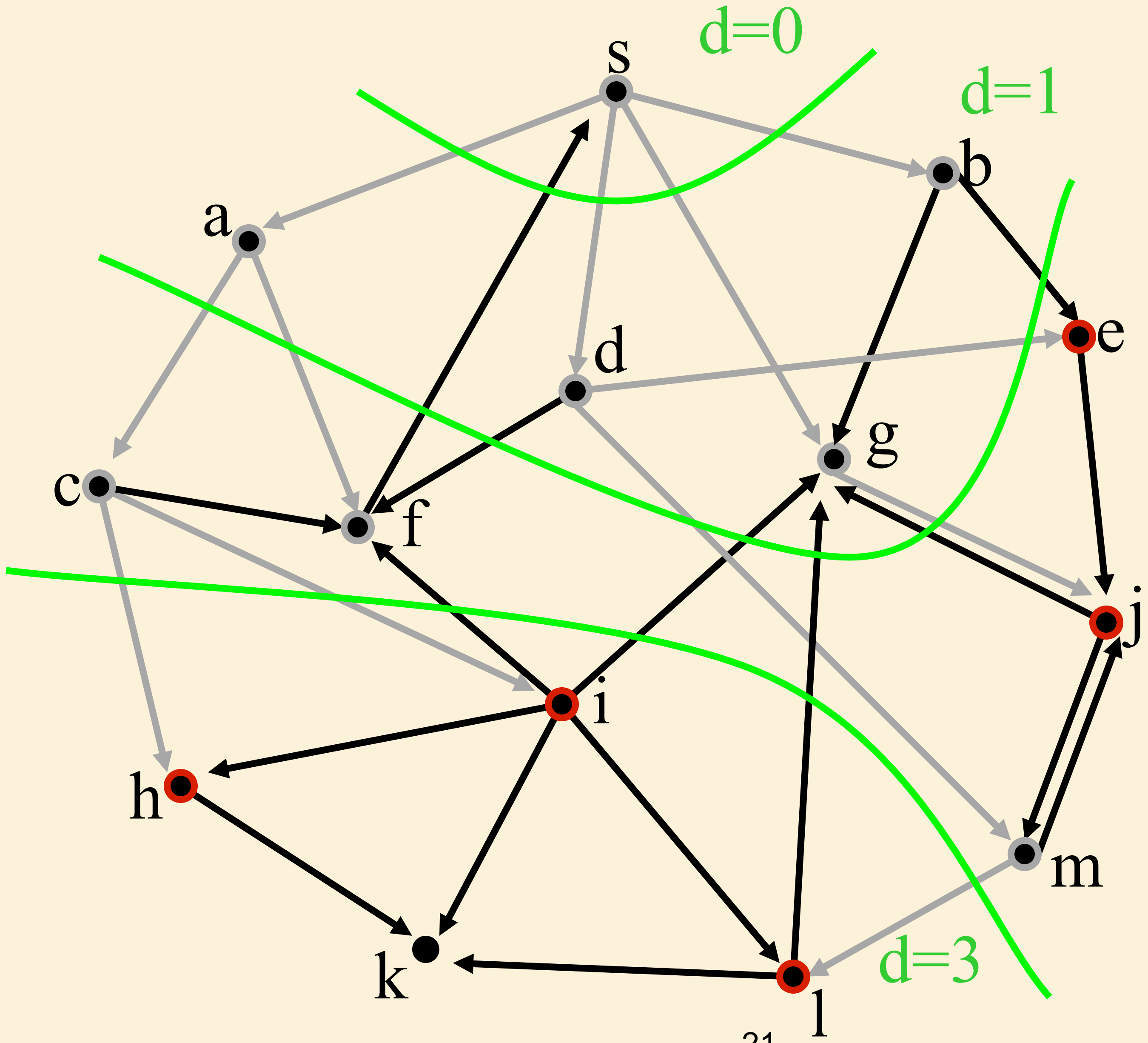
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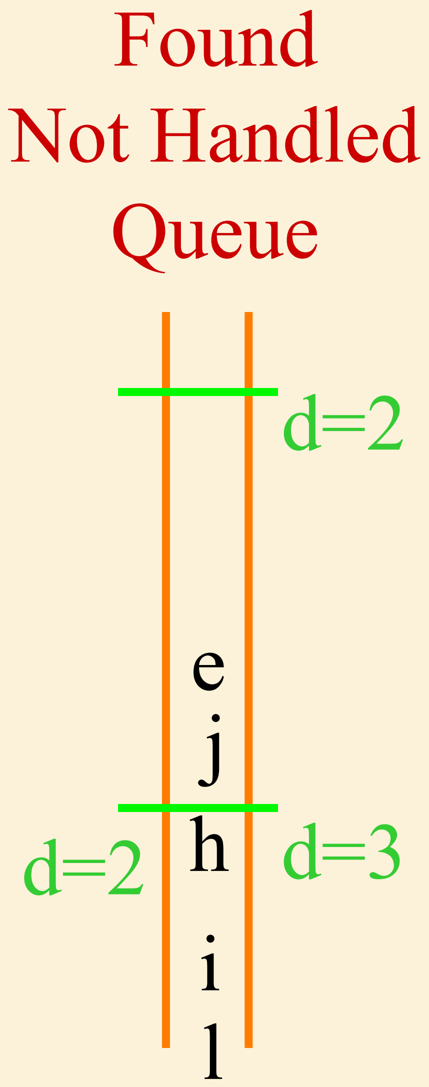
j
k
l
m
d=3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d=2
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d=3
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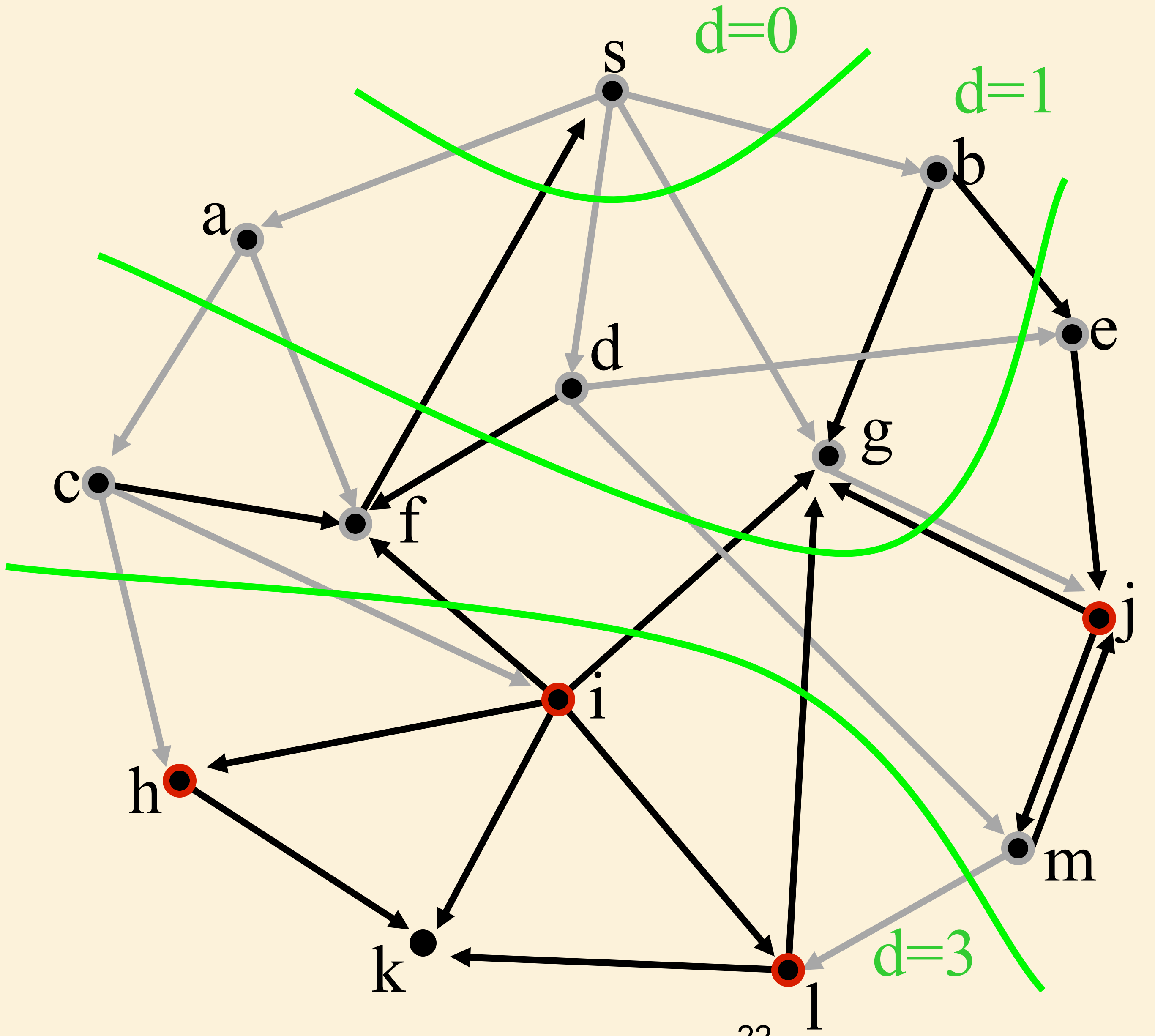
l

BFS



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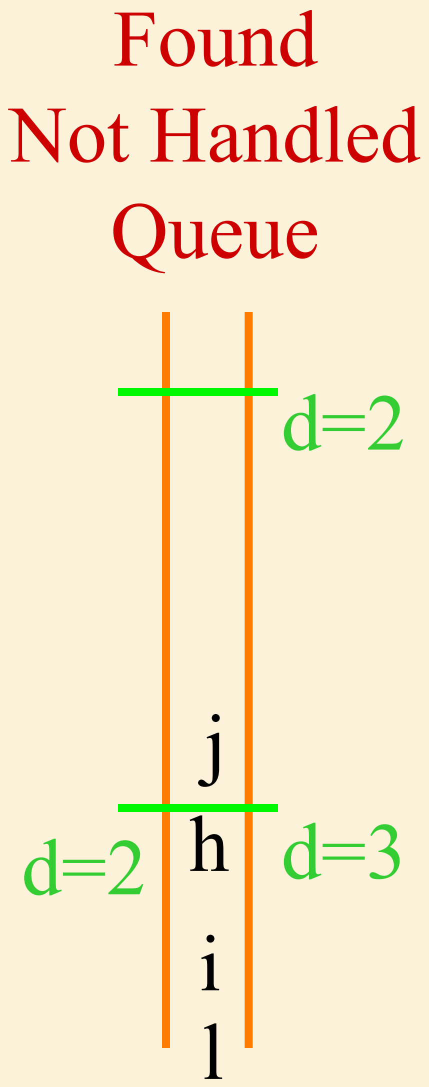
d=3



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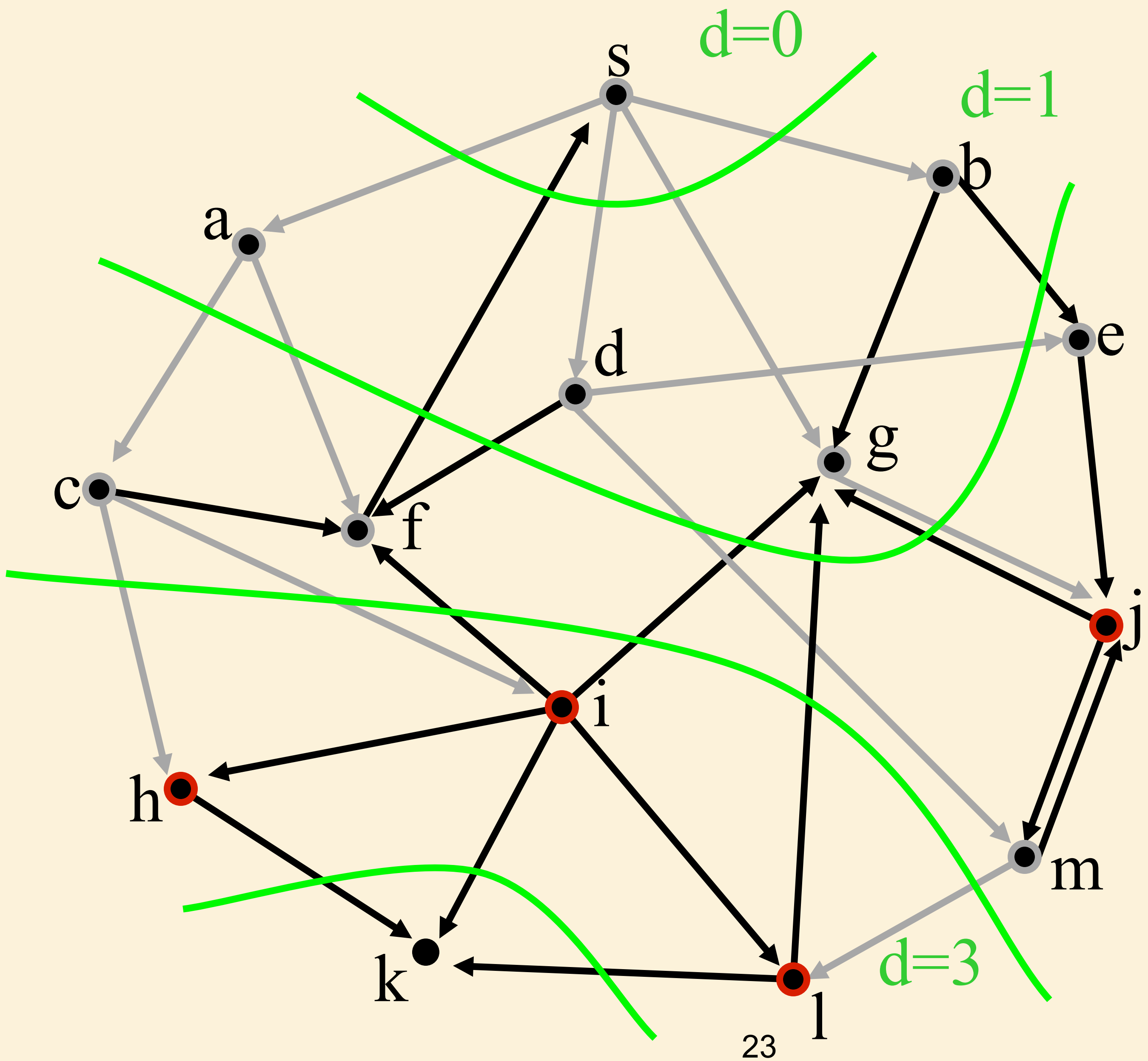
d=2
d=3

BFS



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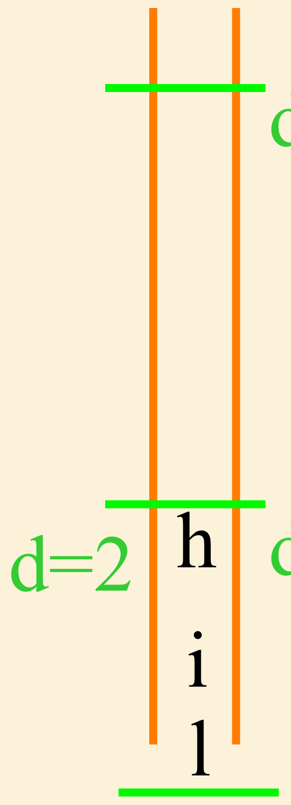

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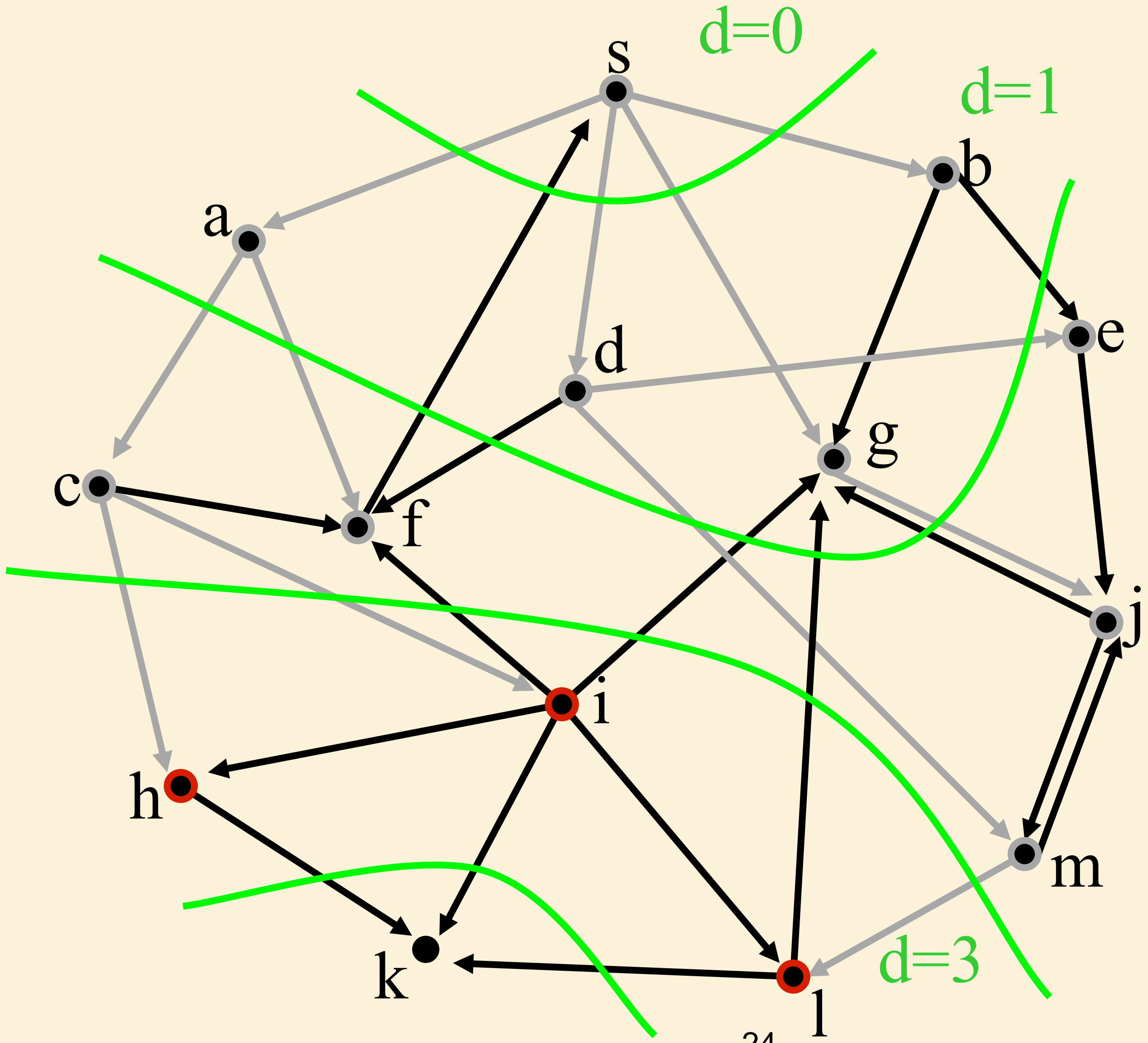

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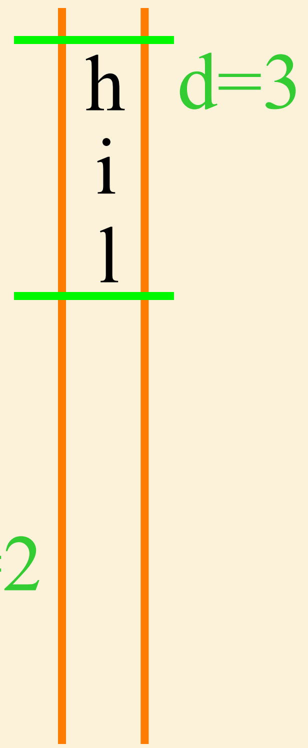

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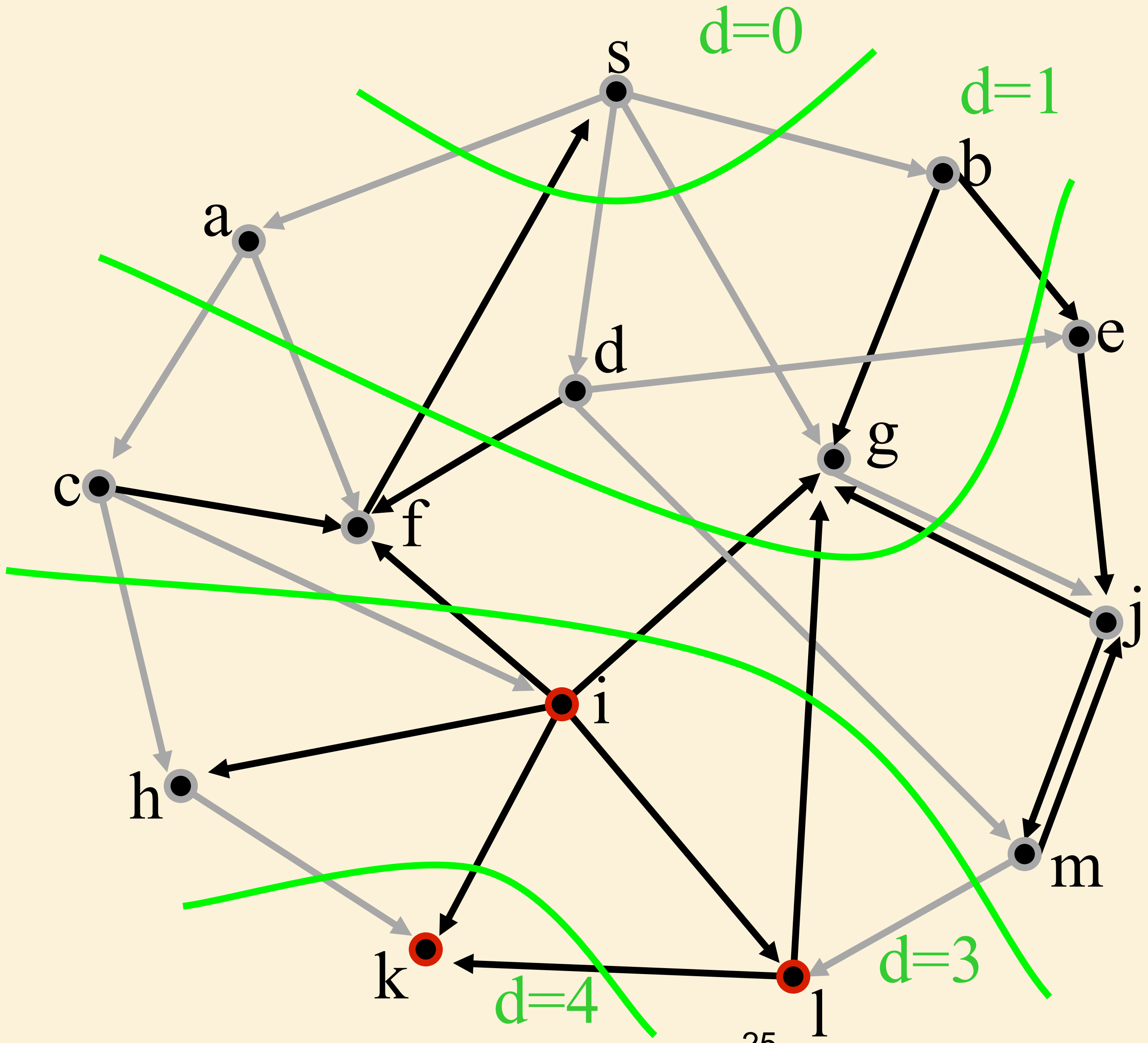

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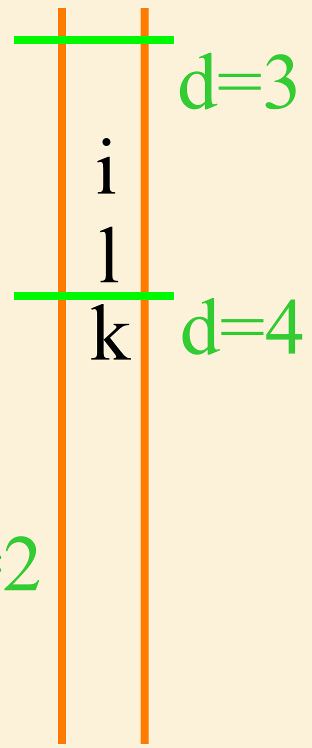

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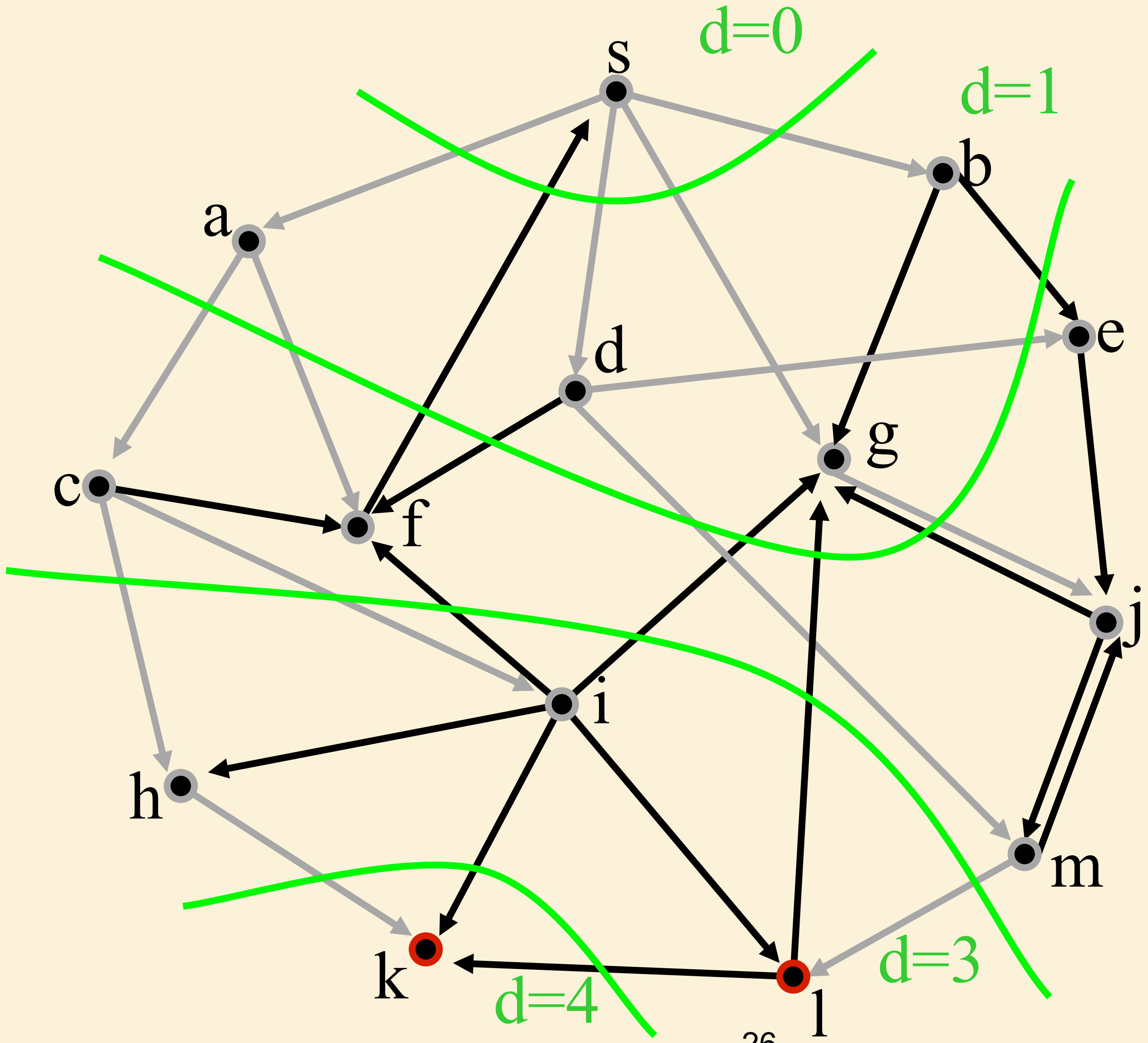

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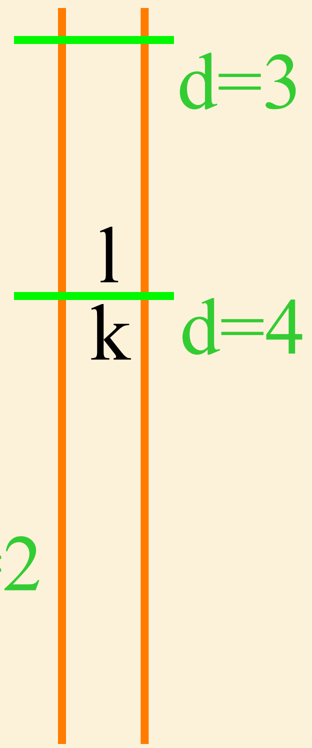

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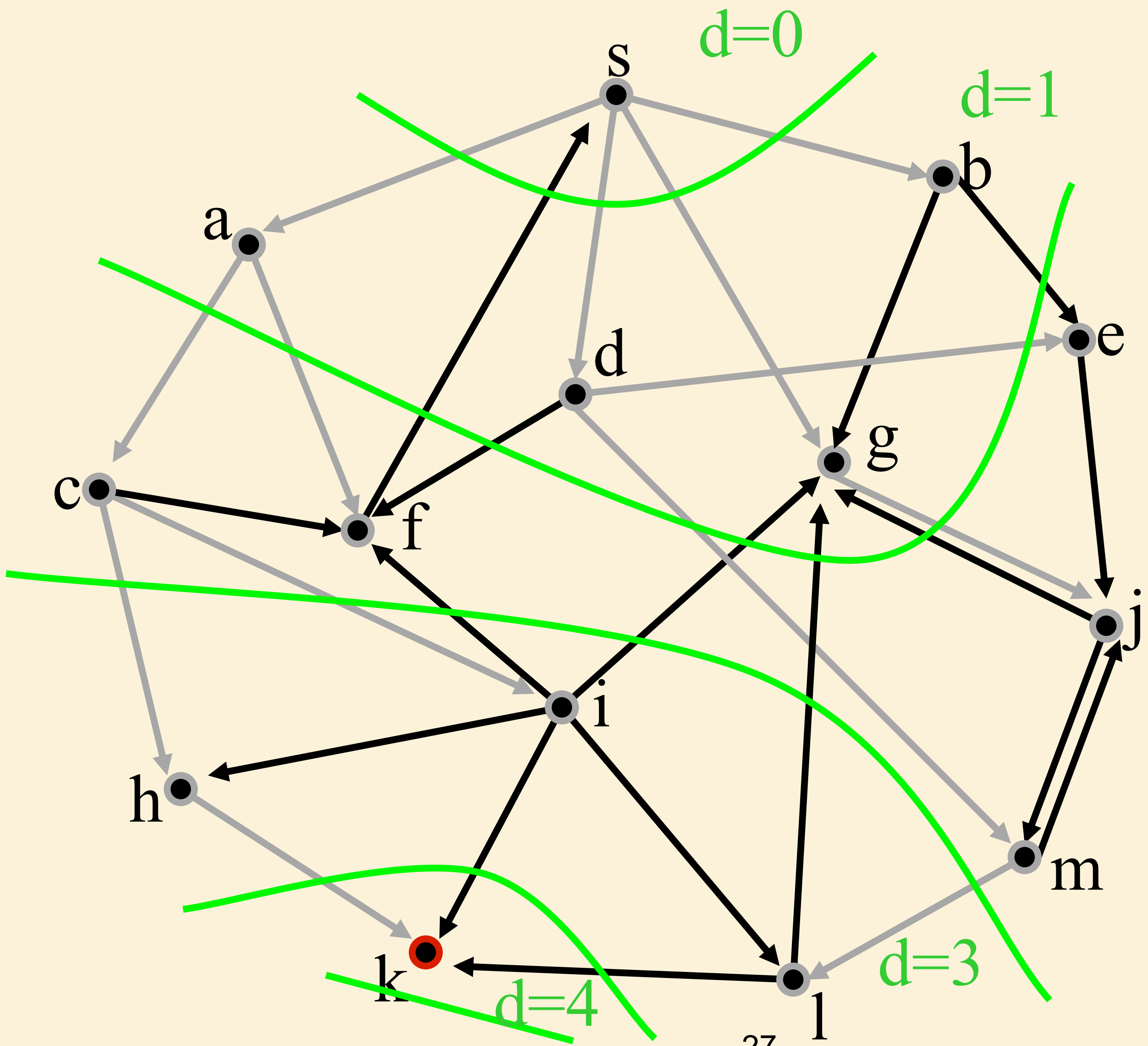

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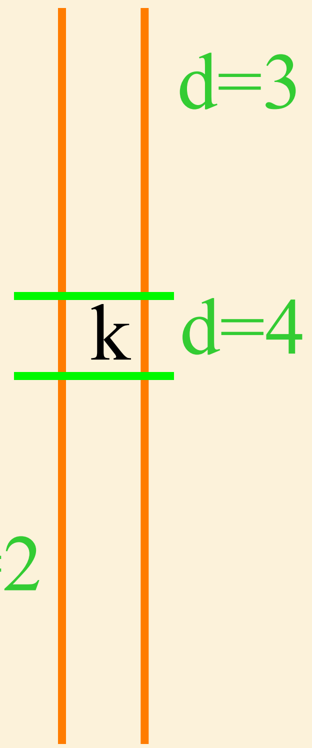

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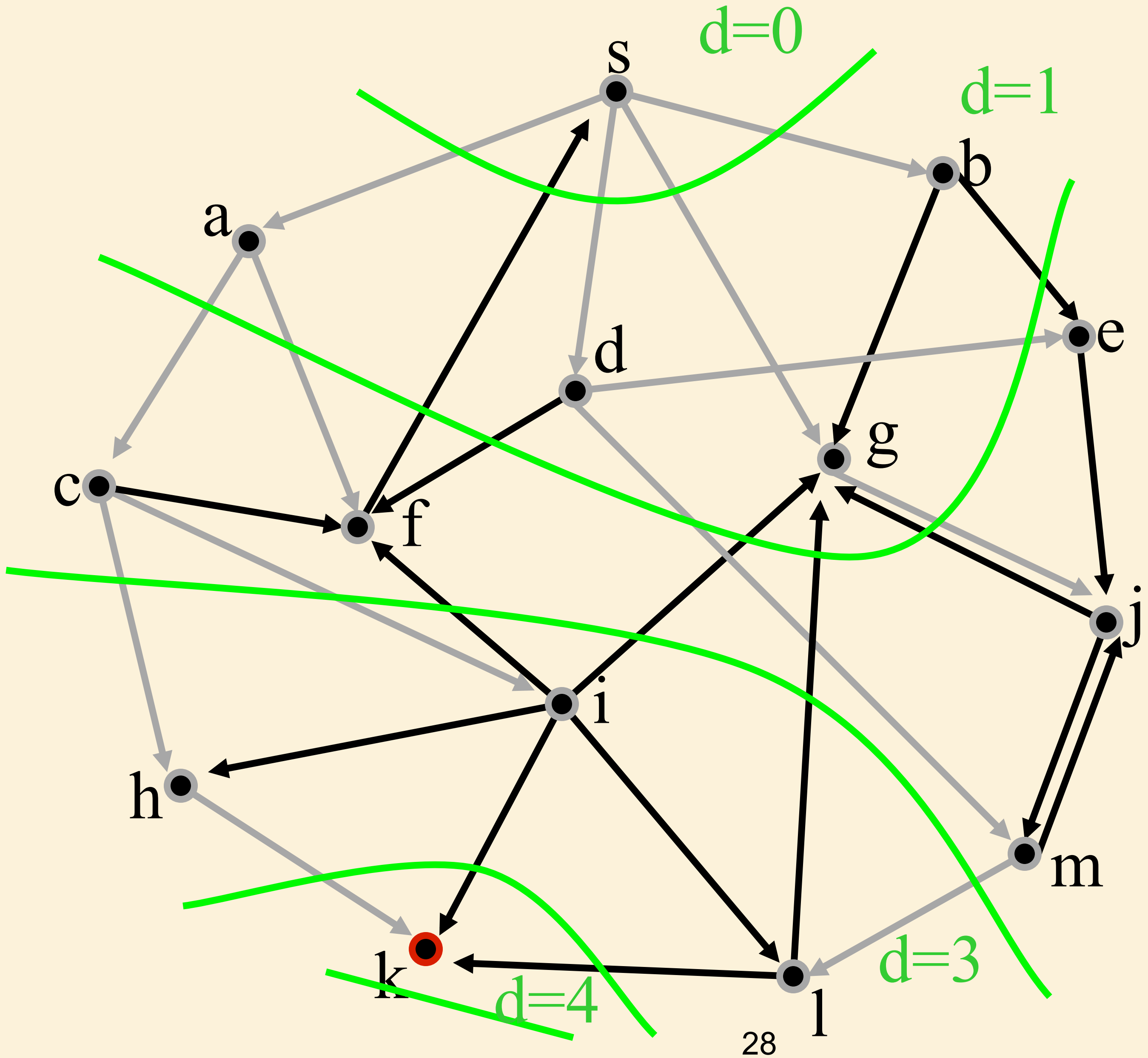

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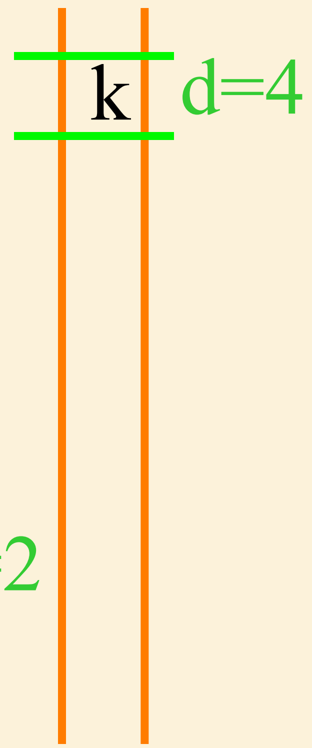
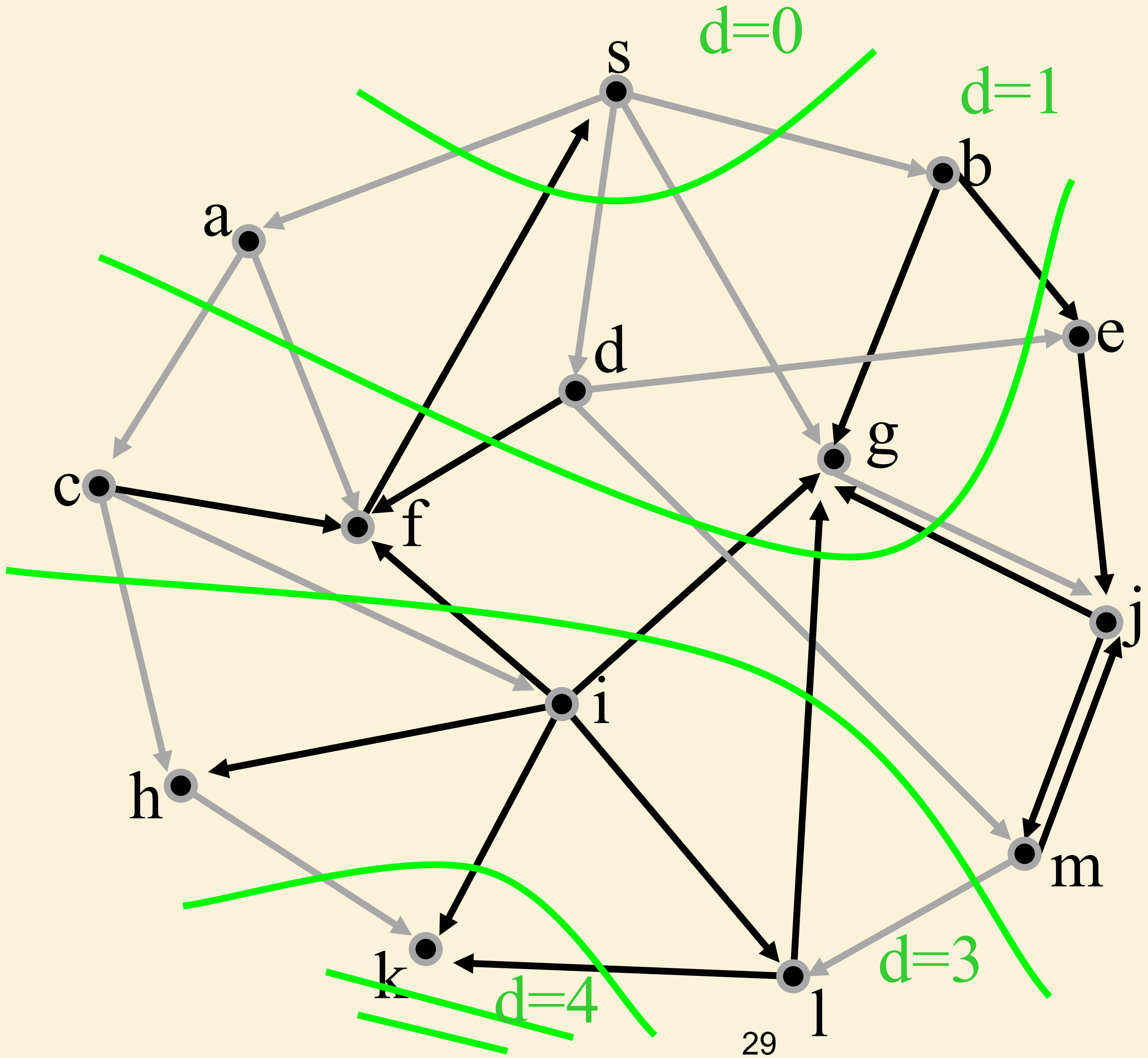


BFS



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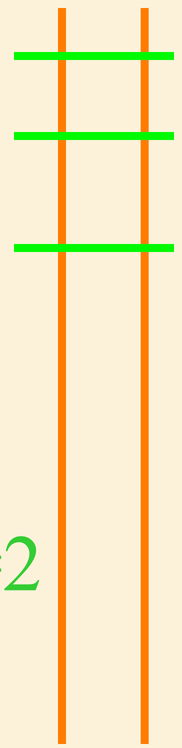


Breadth-First Search Algorithm

```
BFS($G, s$)
for each vertex $u \in V[G] - \{s\}$
    do $color[u] \leftarrow$ BLACK
       $d[u] \leftarrow \infty$
       $\pi[u] \leftarrow$ NIL
$color[s] \leftarrow$ RED
$d[s] \leftarrow 0$
$\pi[s] \leftarrow$ NIL
$Q \leftarrow \emptyset$
ENQUEUE($Q, s$)
while $Q \neq \emptyset$
    do $u \leftarrow$ DEQUEUE($Q$)
       for each $v \in Adj[u]$
           do if $color[v] =$ BLACK
                 then $color[v] \leftarrow$ RED
                      $d[v] \leftarrow d[u] + 1$
                      $\pi[v] \leftarrow u$
                      ENQUEUE($Q, v$)
       $color[u] \leftarrow$ GRAY
```

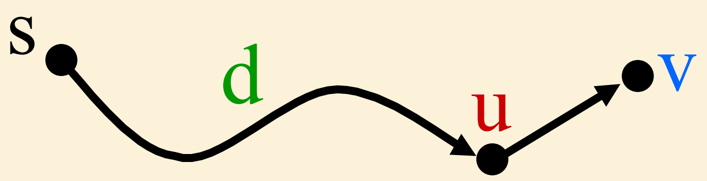
- Q is a FIFO queue.
- Each vertex assigned finite *d* value at most once.
- *Q* contains vertices with d values {*i*, …, *i*, *i*+1, …, *i*+1}
- *d* values assigned are monotonically increasing over time.

Breadth-First-Search is Greedy

- Vertices are handled:
  - in order of their discovery (FIFO queue)
  - Smallest *d* values first

Correctness

Basic Steps:

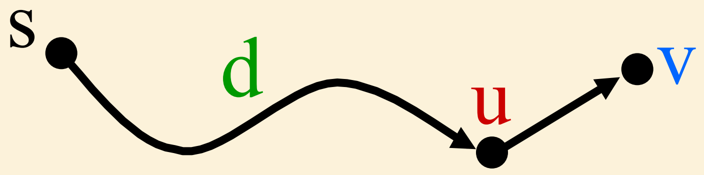


The shortest path to u has length d & there is an edge from u to v

There is a path to v with length d+1.

Correctness

- Vertices are discovered in order of their distance from the source vertex *s*.
- When we discover *v*, how do we know there is not a shorter path to *v*?
  - Because if there was, we would already have discovered it!



Correctness

Input: Graph $G = (V, E)$ (directed or undirected) and source vertex $s \in V$.

Output:

$d[v] =$ distance from s to v, $\forall v \in V$.

$\pi[v] = u$ such that (u,v) is last edge on shortest path from s to v.

Two-step proof:

On exit:

1. $d[v] \geq \delta(s,v) \forall v \in V$

2. $d[v] \not> \delta(s,v) \forall v \in V$

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

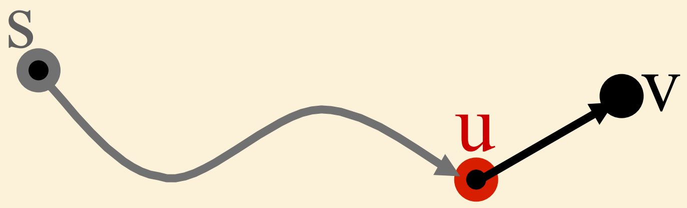
Proof: There exists a path from s to v of length $d[v]$.

By Induction:

Suppose it is true for all vertices thus far discovered (red and grey).

v is discovered from some adjacent vertex u being handled.

$$\begin{aligned} \rightarrow d[v] &= d[u]+1 \\ &\geq \delta(s,u)+1 \\ &\geq \delta(s,v) \end{aligned}$$



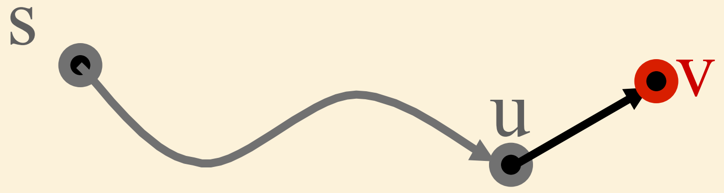
since each vertex v is assigned a d value exactly once,

it follows that on exit, $d[v] \geq \delta(s,v) \forall v \in V$.

Claim 1. *d* is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

Proof: There exists a path from *s* to *v* of length *d*[*v*].

```
BFS(G, s)
 1  for each vertex $u \in V[G] - \{s\}$
 2        do $color[u] \leftarrow$ BLACK
 3           $d[u] \leftarrow \infty$
 4           $\pi[u] \leftarrow$ NIL
 5  $color[s] \leftarrow$ RED
 6  $d[s] \leftarrow 0$
 7  $\pi[s] \leftarrow$ NIL
 8  $Q \leftarrow \emptyset$
 9  ENQUEUE(Q, s)
10  while $Q \neq \emptyset$      $\leftarrow$ <LI>: $d[v] \geq \delta(s,v) \forall$ 'discovered' (red or grey) $v \in V$
11        do $u \leftarrow$ DEQUEUE(Q)
12           for each $v \in Adj[u]$
13               do if $color[v] =$ BLACK
14                     then $color[v] \leftarrow$ RED
15                          $d[v] \leftarrow d[u] + 1$      $\geq \delta(s,u) + 1 \geq \delta(s,v)$
16                          $\pi[v] \leftarrow u$
17                          ENQUEUE(Q, v)
18           $color[u] \leftarrow$ GRAY
```



Claim 2. d is never too big: $d[v] \le \delta(s,v) \forall v \in V$

Proof by contradiction:

Suppose one or more vertices receive a d value greater than δ.

Let v be the vertex with minimum $\delta(s,v)$ that receives such a d value.

Suppose that v is discovered and assigned this d value when vertex x is dequeued.

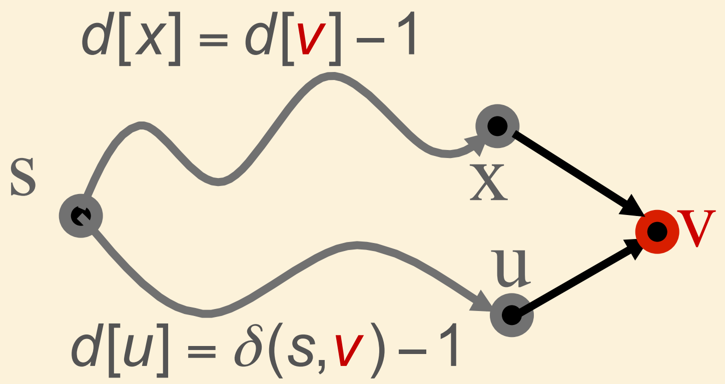
Let u be v's predecessor on a shortest path from s to v.

Then

$$\delta(s,v) < d[v]$$

$$\rightarrow \delta(s,v) - 1 < d[v] - 1$$

$$\rightarrow d[u] < d[x]$$



Recall: vertices are dequeued in increasing order of d value.

$\rightarrow$ u was dequeued before x.

$\rightarrow d[v] = d[u] + 1 = \delta(s,v)$ **Contradiction!**

Correctness

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

Claim 2. d is never too big: $d[v] \leq \delta(s,v) \forall v \in V$

$\Rightarrow d$ is just right: $d[v] = \delta(s,v) \forall v \in V$

Progress?

- On every iteration one vertex is processed (turns gray).

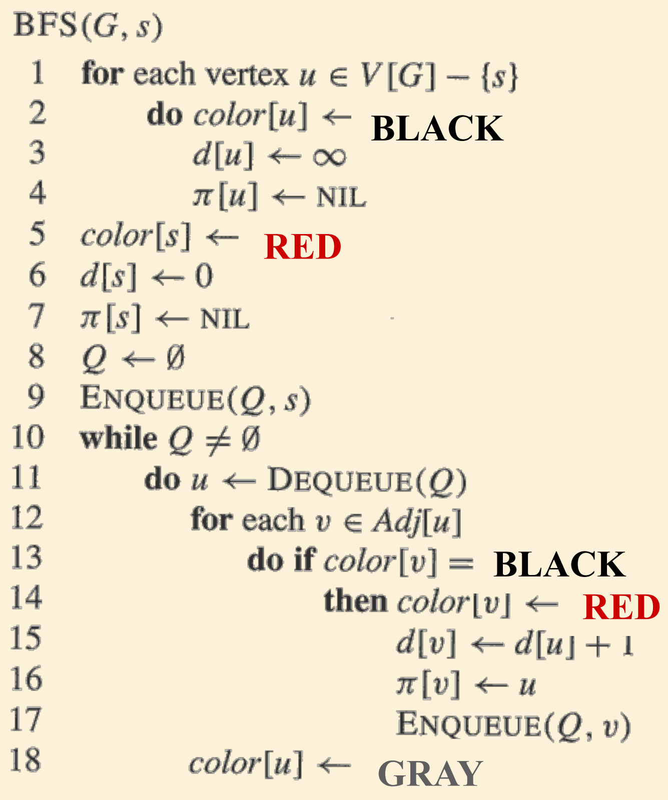
```
BFS($G, s$)
for each vertex $u \in V[G] - \{s\}$
    do $color[u] \leftarrow$ BLACK
       $d[u] \leftarrow \infty$
       $\pi[u] \leftarrow$ NIL
$color[s] \leftarrow$ RED
$d[s] \leftarrow 0$
$\pi[s] \leftarrow$ NIL
$Q \leftarrow \emptyset$
ENQUEUE($Q, s$)
while $Q \neq \emptyset$
    do $u \leftarrow$ DEQUEUE($Q$)
       for each $v \in Adj[u]$
           do if $color[v] =$ BLACK
                then $color[v] \leftarrow$ RED
                     $d[v] \leftarrow d[u] + 1$
                     $\pi[v] \leftarrow u$
                     ENQUEUE($Q, v$)
       $color[u] \leftarrow$ GRAY
```

Running Time

Each vertex is enqueued at most once $\rightarrow O(V)$

Each entry in the adjacency lists is scanned at most once $\rightarrow$ O(E)

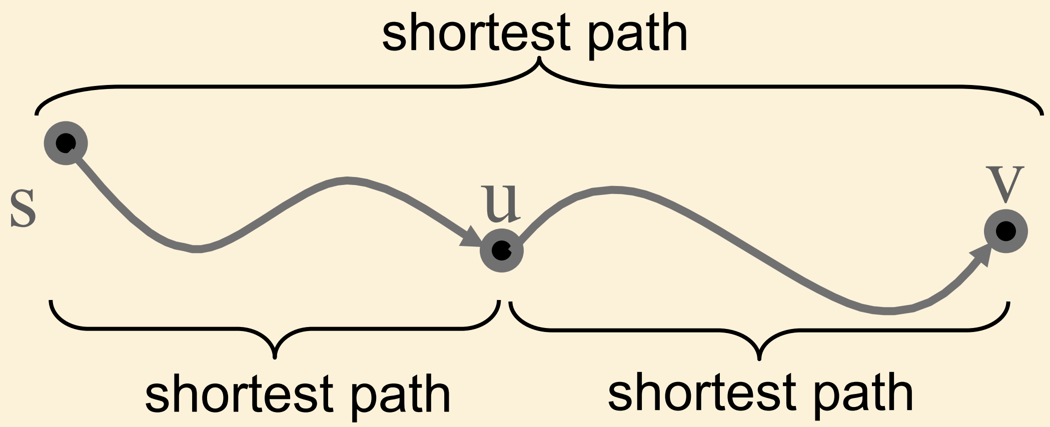
Thus run time is $O(V + E)$.



```
BFS($G, s$)
for each vertex $u \in V[G] - \{s\}$
    do $color[u] \leftarrow$ BLACK
       $d[u] \leftarrow \infty$
       $\pi[u] \leftarrow$ NIL
$color[s] \leftarrow$ RED
$d[s] \leftarrow 0$
$\pi[s] \leftarrow$ NIL
$Q \leftarrow \emptyset$
ENQUEUE($Q, s$)
while $Q \neq \emptyset$
    do $u \leftarrow$ DEQUEUE($Q$)
       for each $v \in Adj[u]$
           do if $color[v] =$ BLACK
                 then $color[v] \leftarrow$ RED
                      $d[v] \leftarrow d[u] + 1$
                      $\pi[v] \leftarrow u$
                      ENQUEUE($Q, v$)
       $color[u] \leftarrow$ GRAY
```

Optimal Substructure Property

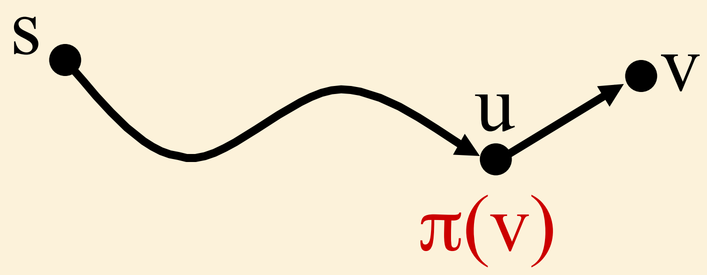
- The shortest path problem has the optimal substructure property:
  - Every subpath of a shortest path is a shortest path.

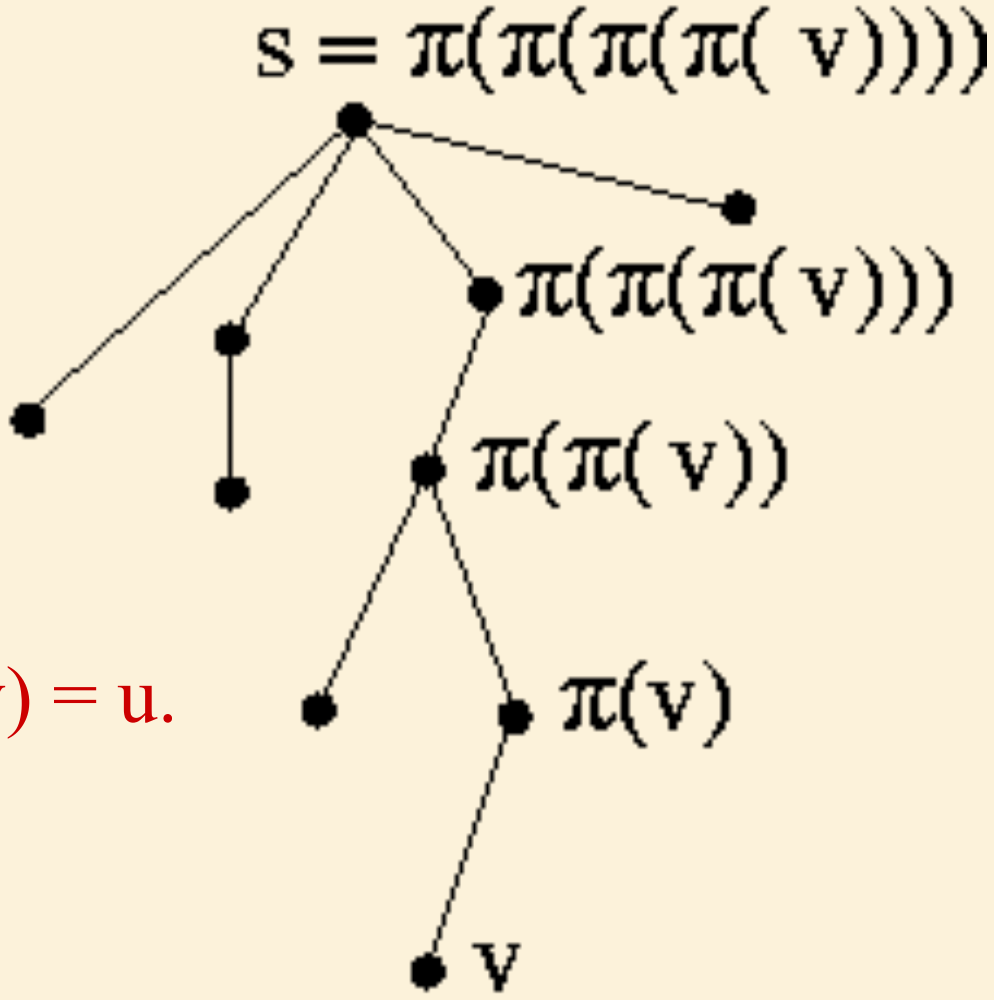


- The optimal substructure property
  - is a hallmark of both greedy and dynamic programming algorithms.
  - allows us to compute both shortest path distance and the shortest paths themselves by storing only one *d* value and one predecessor value per vertex.

Recovering the Shortest Path

For each node v, store predecessor of v in π(v).

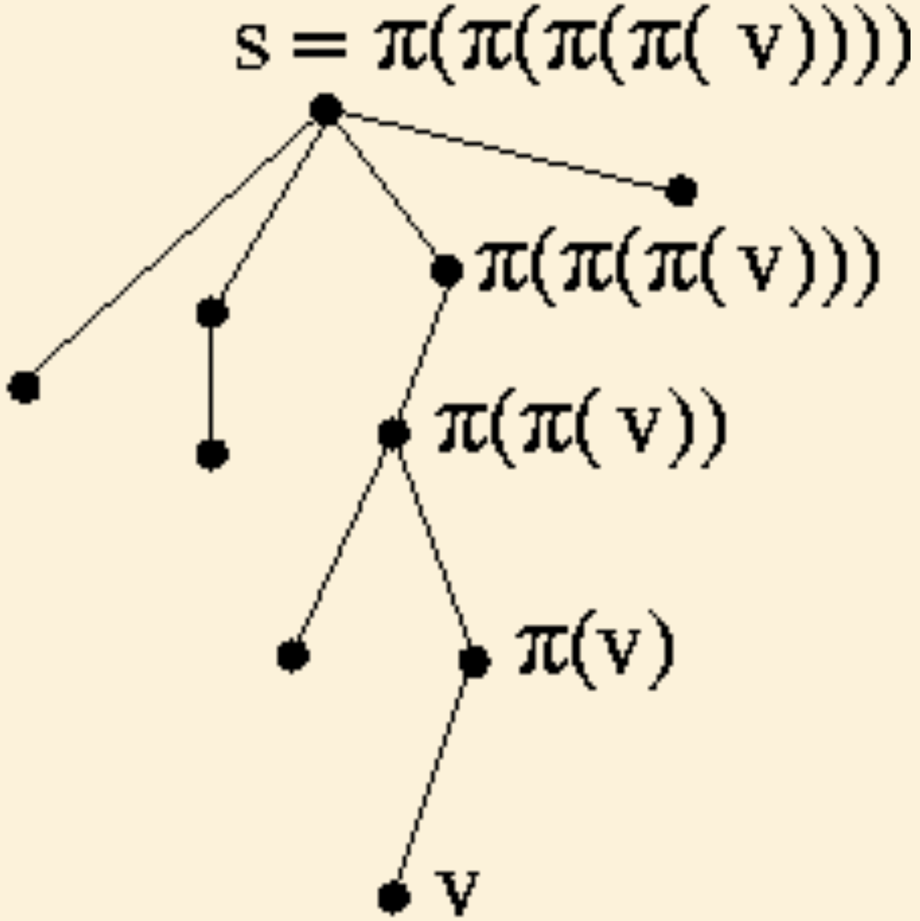




Predecessor of v is π(v) = u.

Recovering the Shortest Path

```
PRINT-PATH($G$, $s$, $v$)
Precondition:  $s$ and $v$ are vertices of graph $G$
Postcondition: the vertices on the shortest path from $s$ to $v$ have been printed in order
if $v = s$ then
    print $s$
else if $\pi[v]$ = NIL then
    print "no path from" $s$ "to" $v$ "exists"
else
    PRINT-PATH($G$, $s$, $\pi[v]$)
    print $v$
```



Colours are actually not required

```
BFS$(V, E, s)$
for each $u \in V - \{s\}$
    do $d[u] \leftarrow \infty$
$d[s] \leftarrow 0$
$Q \leftarrow \emptyset$
ENQUEUE$(Q, s)$
while $Q \neq \emptyset$
    do $u \leftarrow$ DEQUEUE$(Q)$
       for each $v \in Adj[u]$
           do if $d[v] = \infty$
                 then $d[v] \leftarrow d[u] + 1$
                      ENQUEUE$(Q, v)$
```

Depth First Search (DFS)

- Idea:
  - Continue searching “deeper” into the graph, until we get stuck.
  - If all the edges leaving *v* have been explored we “backtrack” to the vertex from which *v* was discovered.
- Does not recover shortest paths, but can be useful for extracting other properties of graph, e.g.,
  - Topological sorts
  - Detection of cycles
  - Extraction of strongly connected components

Depth-First Search

Input: Graph $G = (V, E)$ (directed or undirected)

Output: 2 timestamps on each vertex:

$d[v]$ = discovery time.

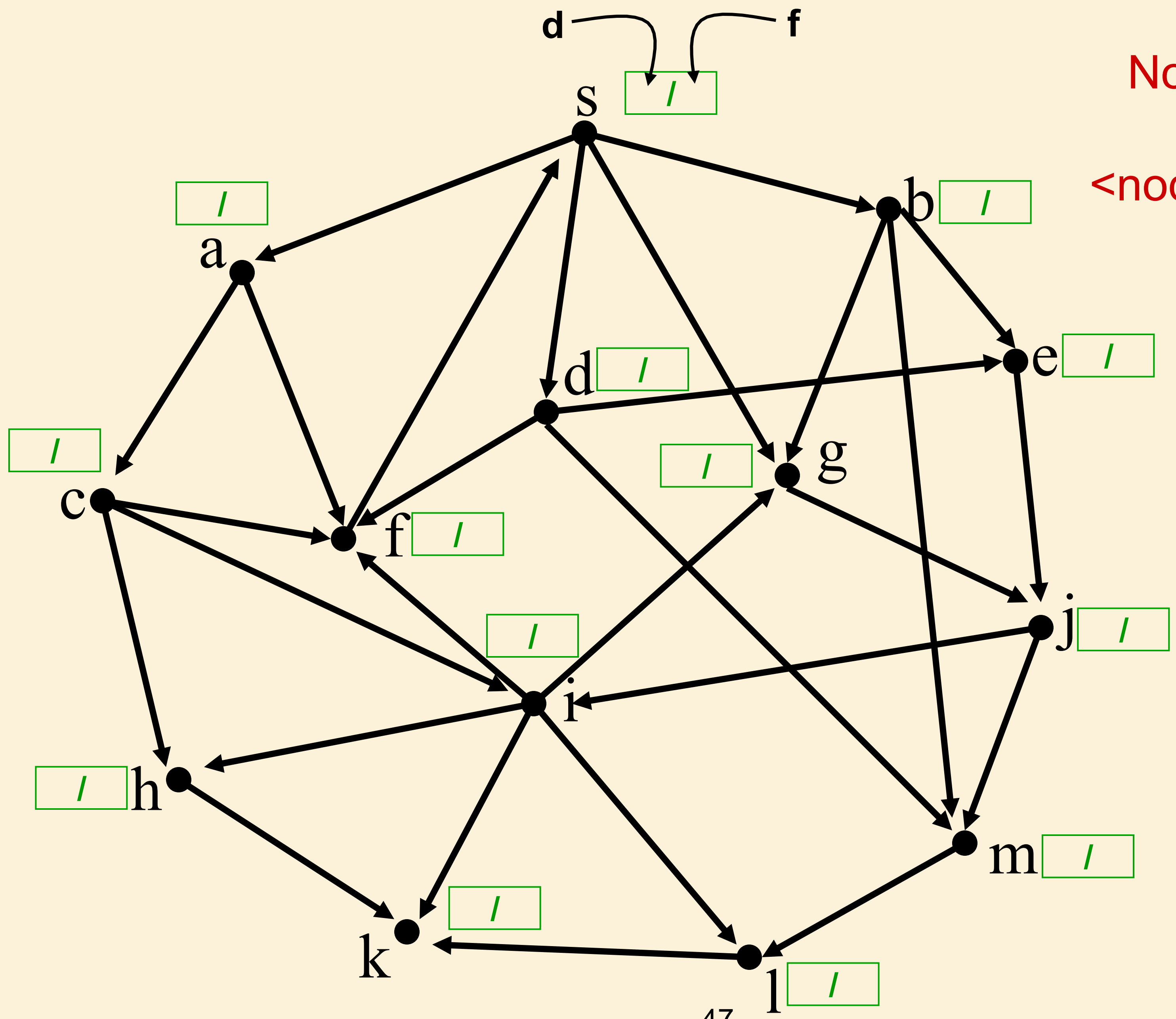
$f[v]$ = finishing time.

$$1 \le d[v] < f[v] \le 2|V|$$

- Explore *every* edge, starting from different vertices if necessary.
- As soon as vertex discovered, explore from it.
- Keep track of progress by colouring vertices:
  - Black: undiscovered vertices
  - Red: discovered, but not finished (still exploring from it)
  - Gray: finished (found everything reachable from it).

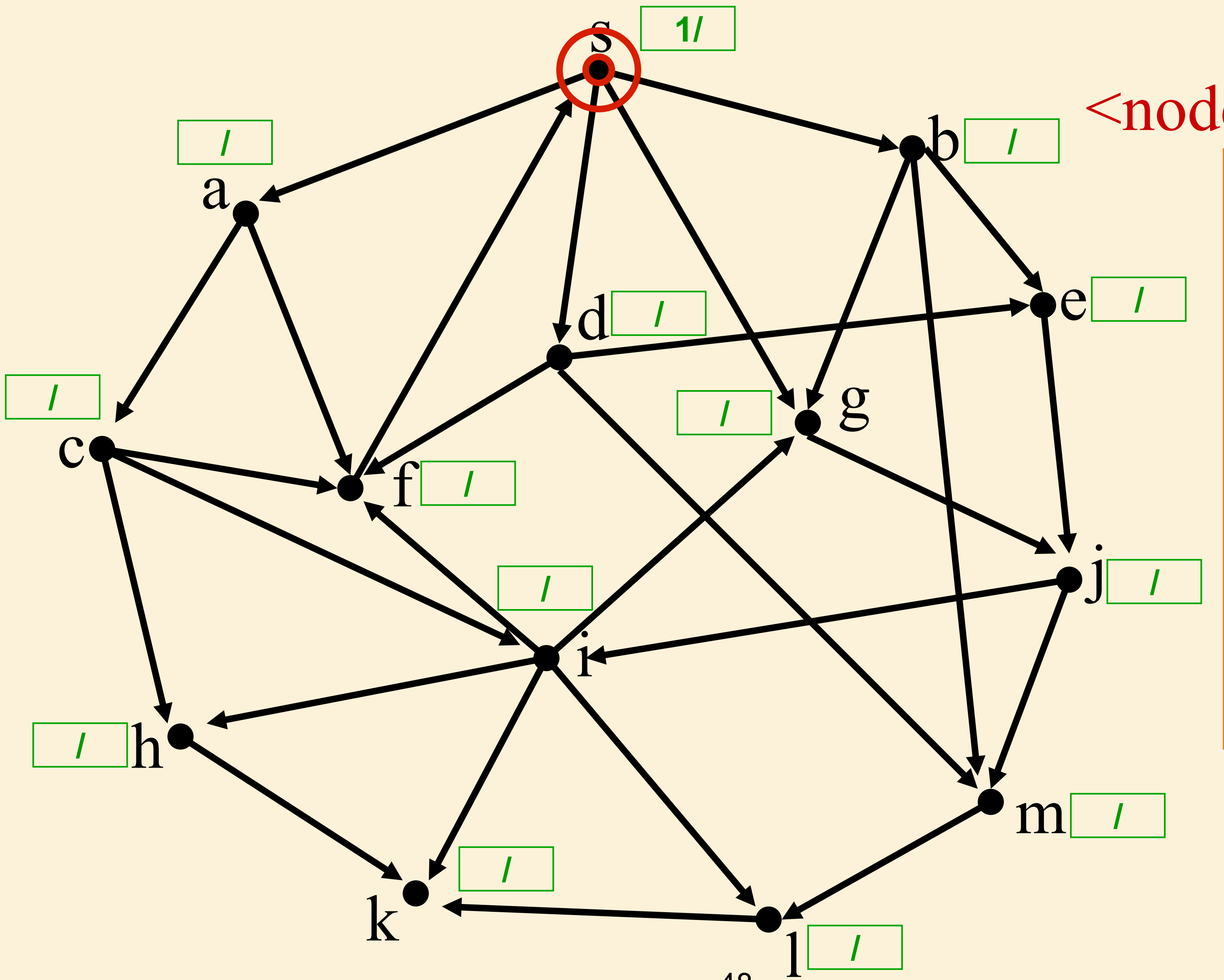
DFS

Note: Stack is Last-In First-Out (LIFO)



Found
Not Handled
Stack
<node,# edges>

DFS



Found
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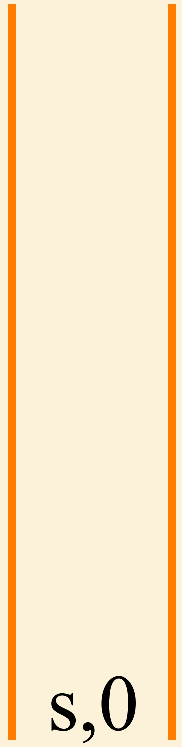
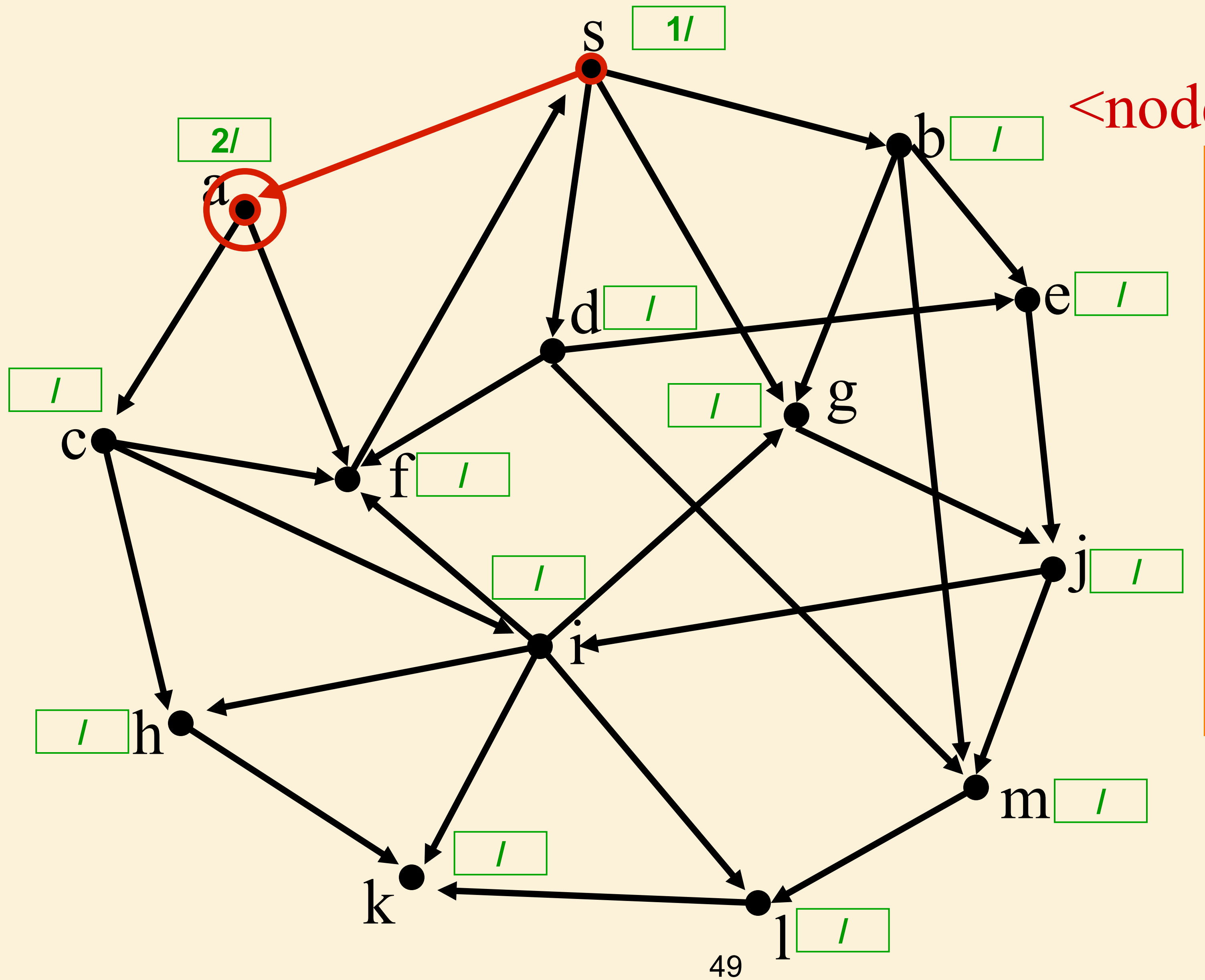
Stack
<node,# edges>



DFS



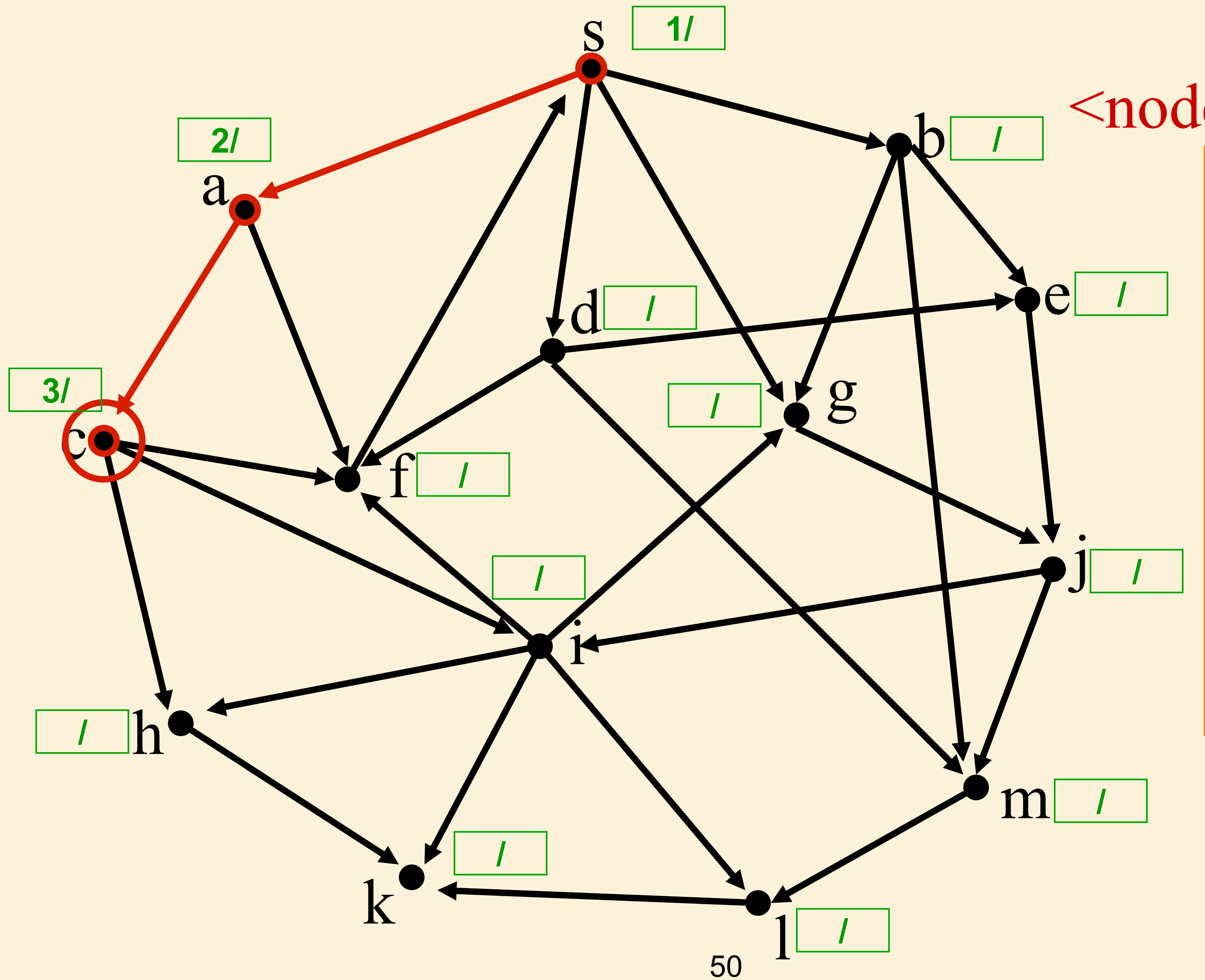
Found
Not Handled
Stack
<node,# edges>

a,0
s,1

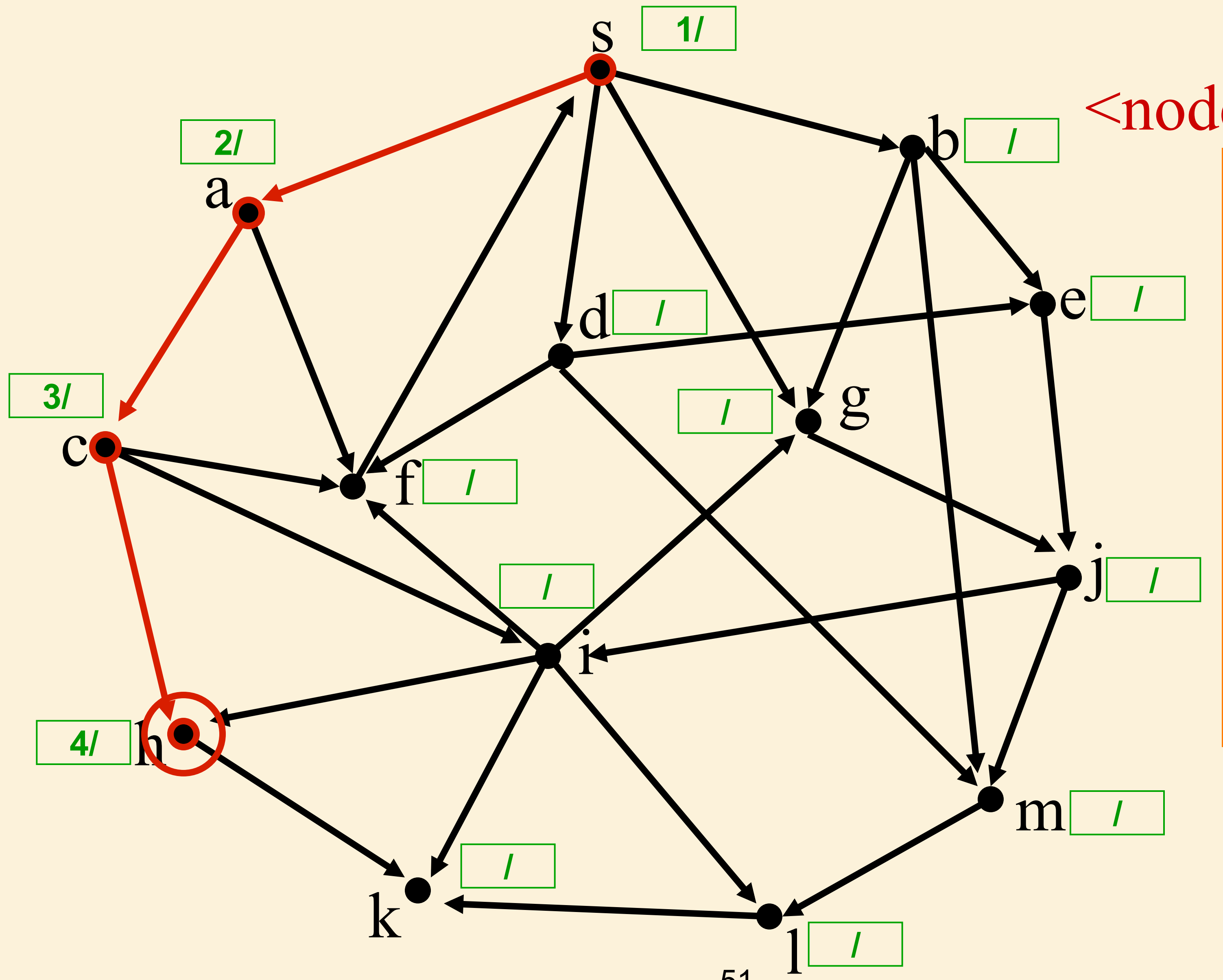
DFS

Found
Not Handled
Stack
<node,# edges>

c,0
a,1
s,1



DFS



Found
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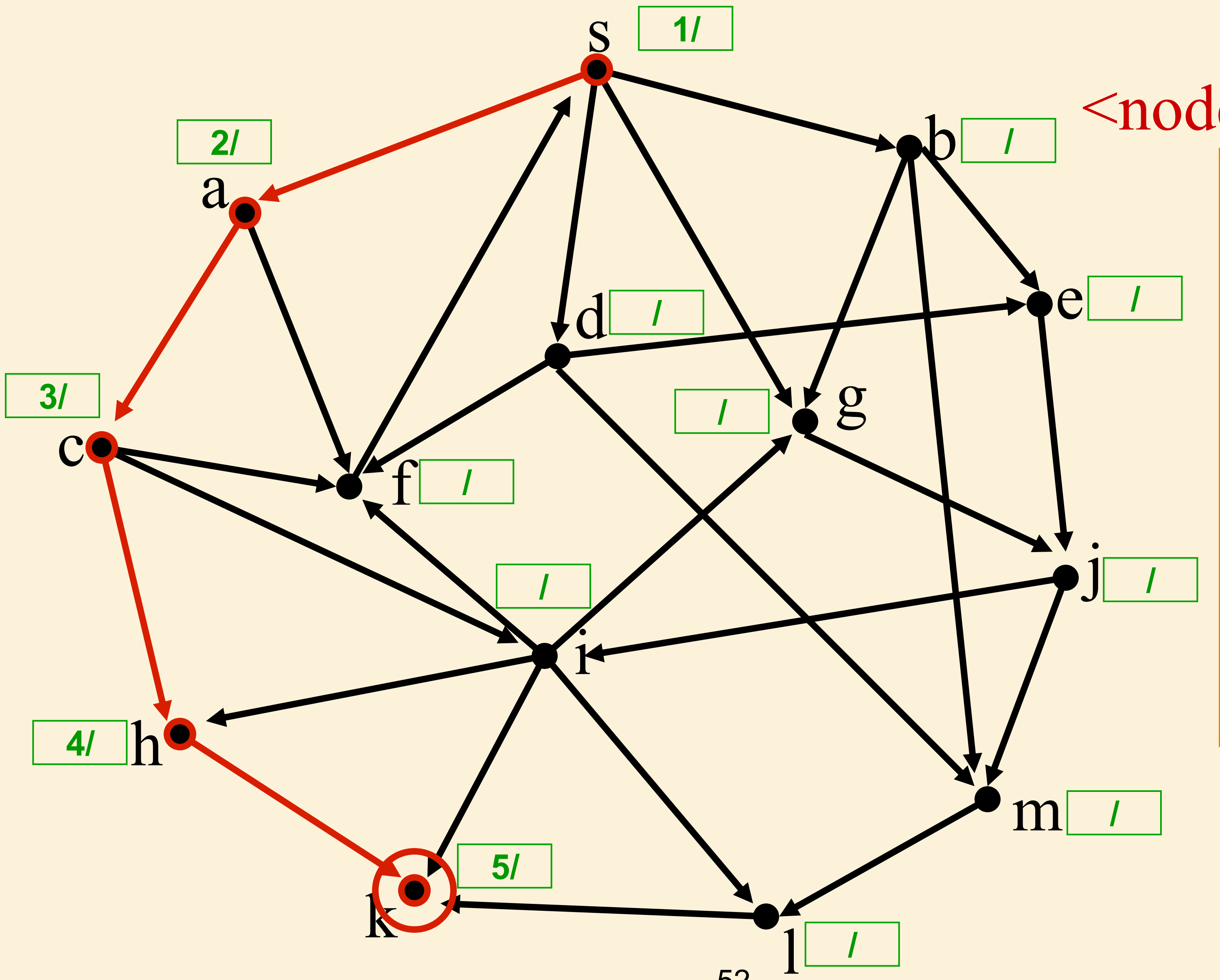
Stack
<node,# edges>

h,0
c,1
a,1
s,1

DFS



Found
Not Handled
Stack
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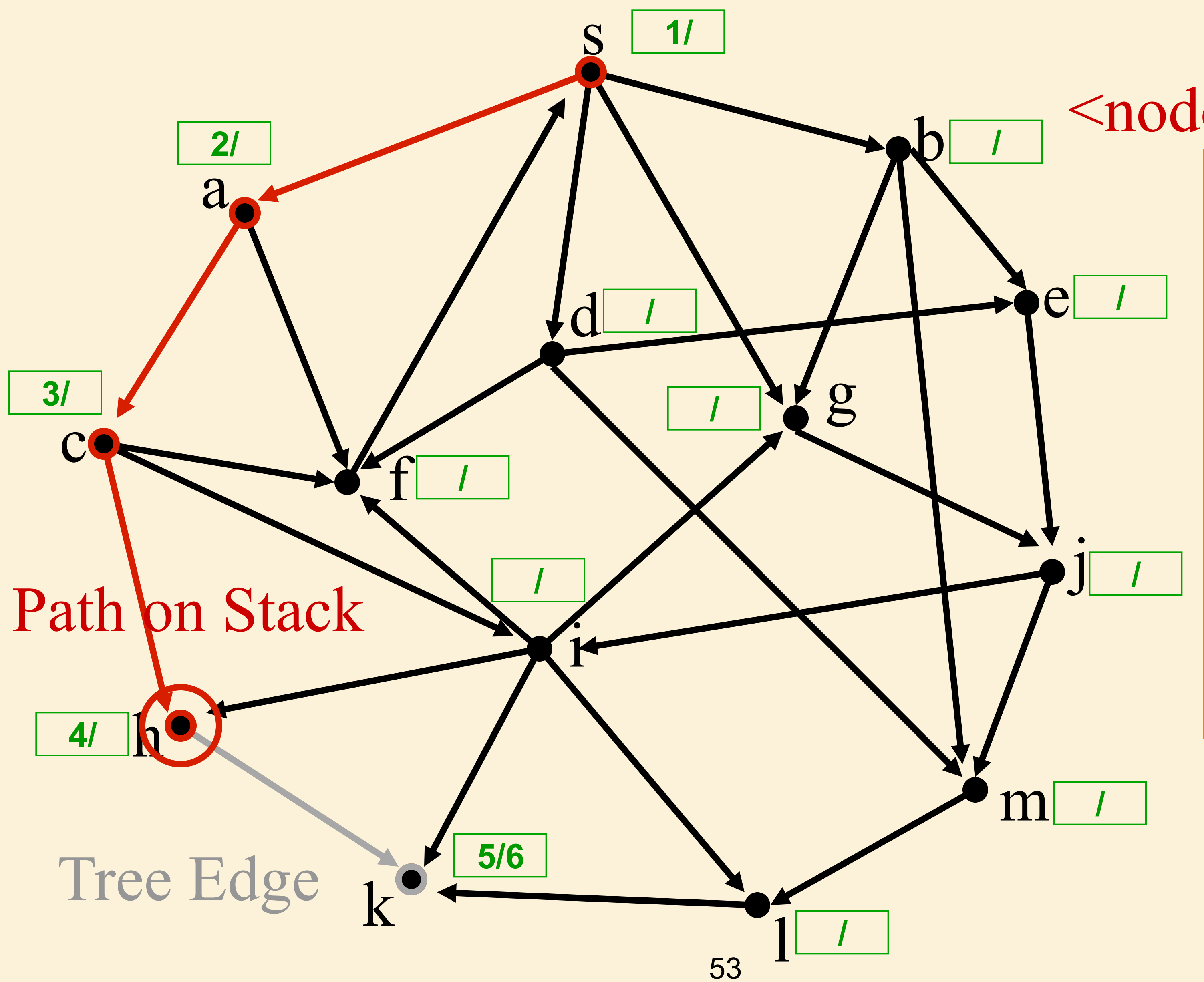
k,0
h,1
c,1
a,1
s,1

DFS



Found
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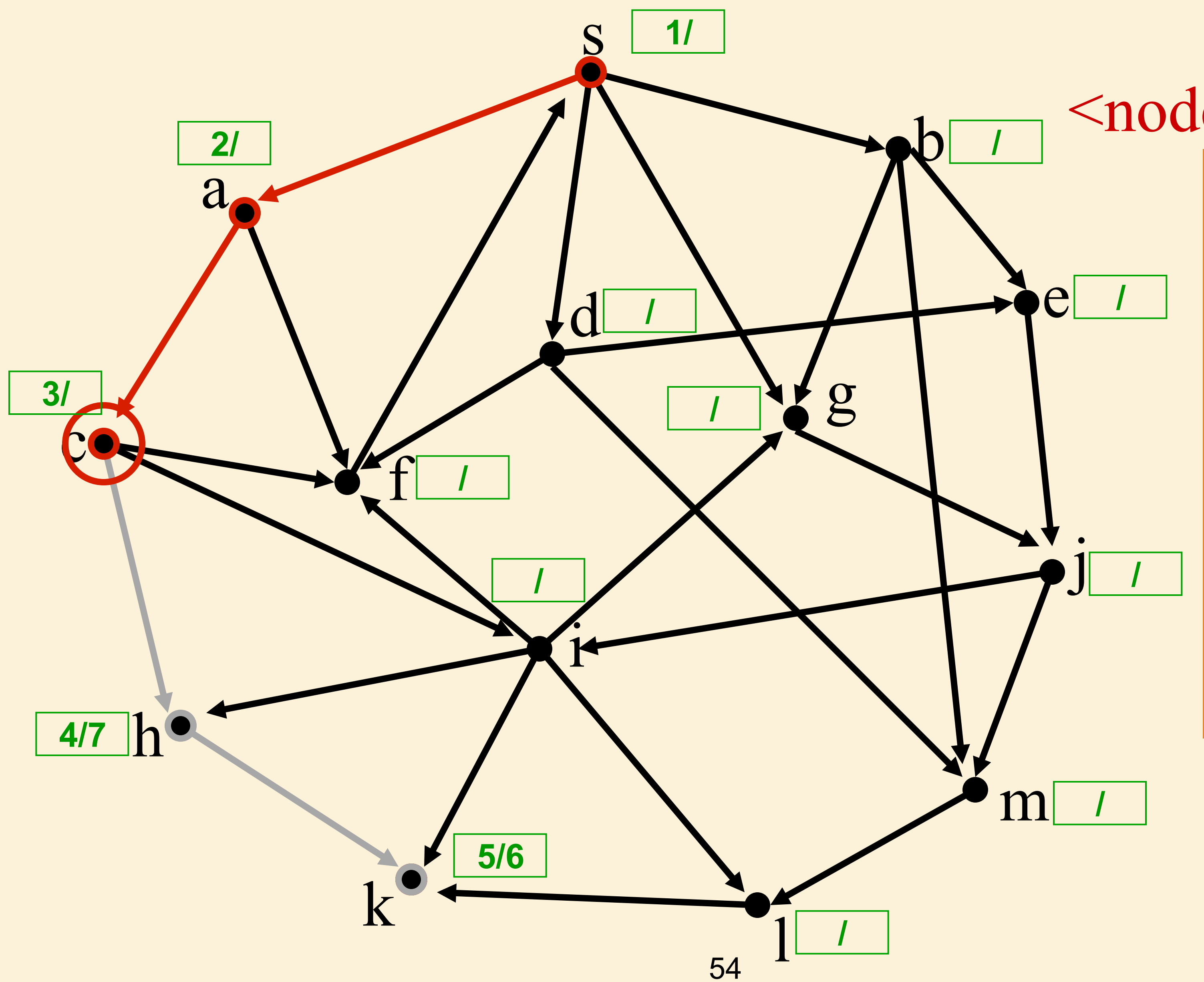
Stack
<node,# edges>

h,1
c,1
a,1
s,1

DFS



Found
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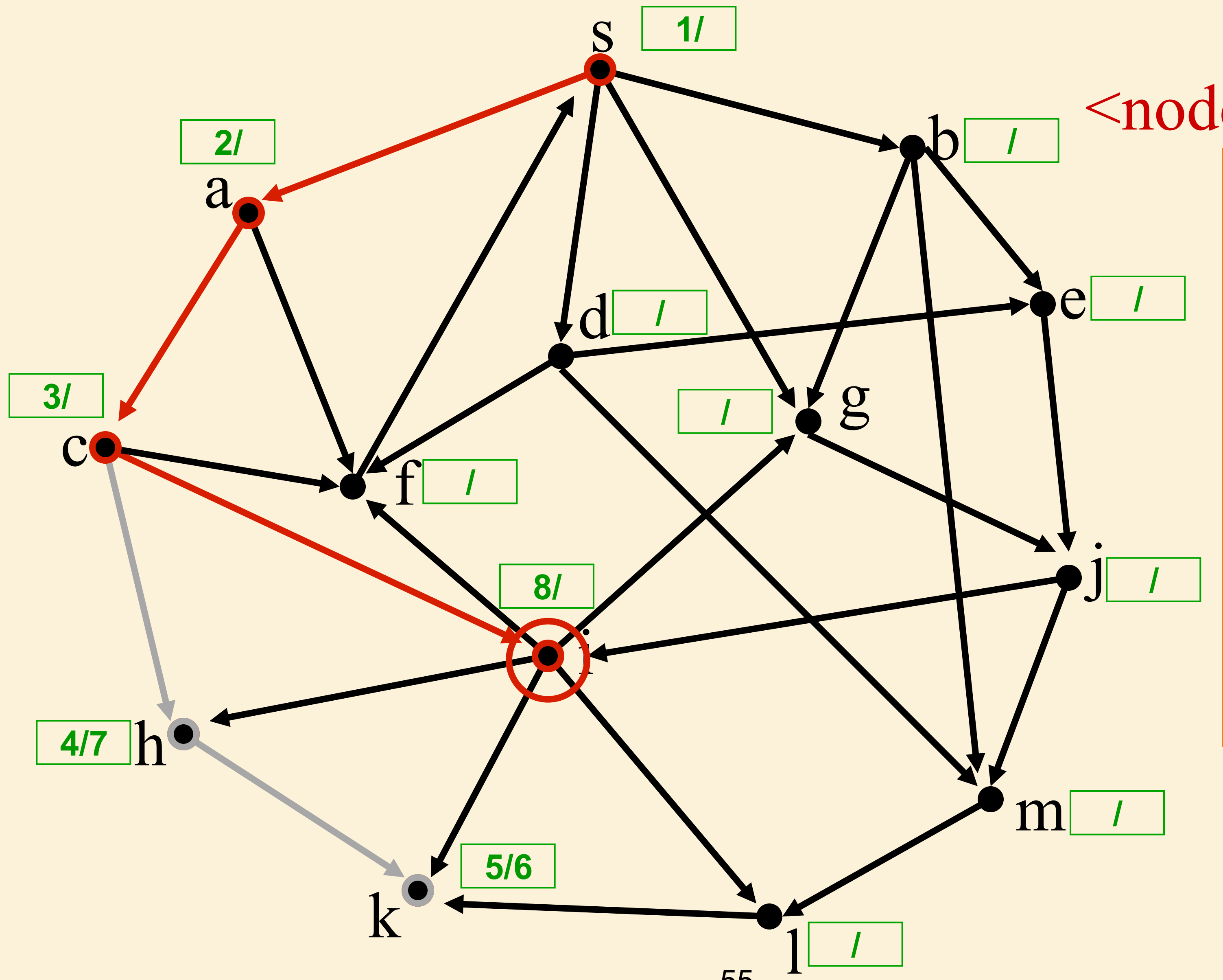
Stack
<node,# edges>

c, 1
a, 1
s, 1

DFS



Found
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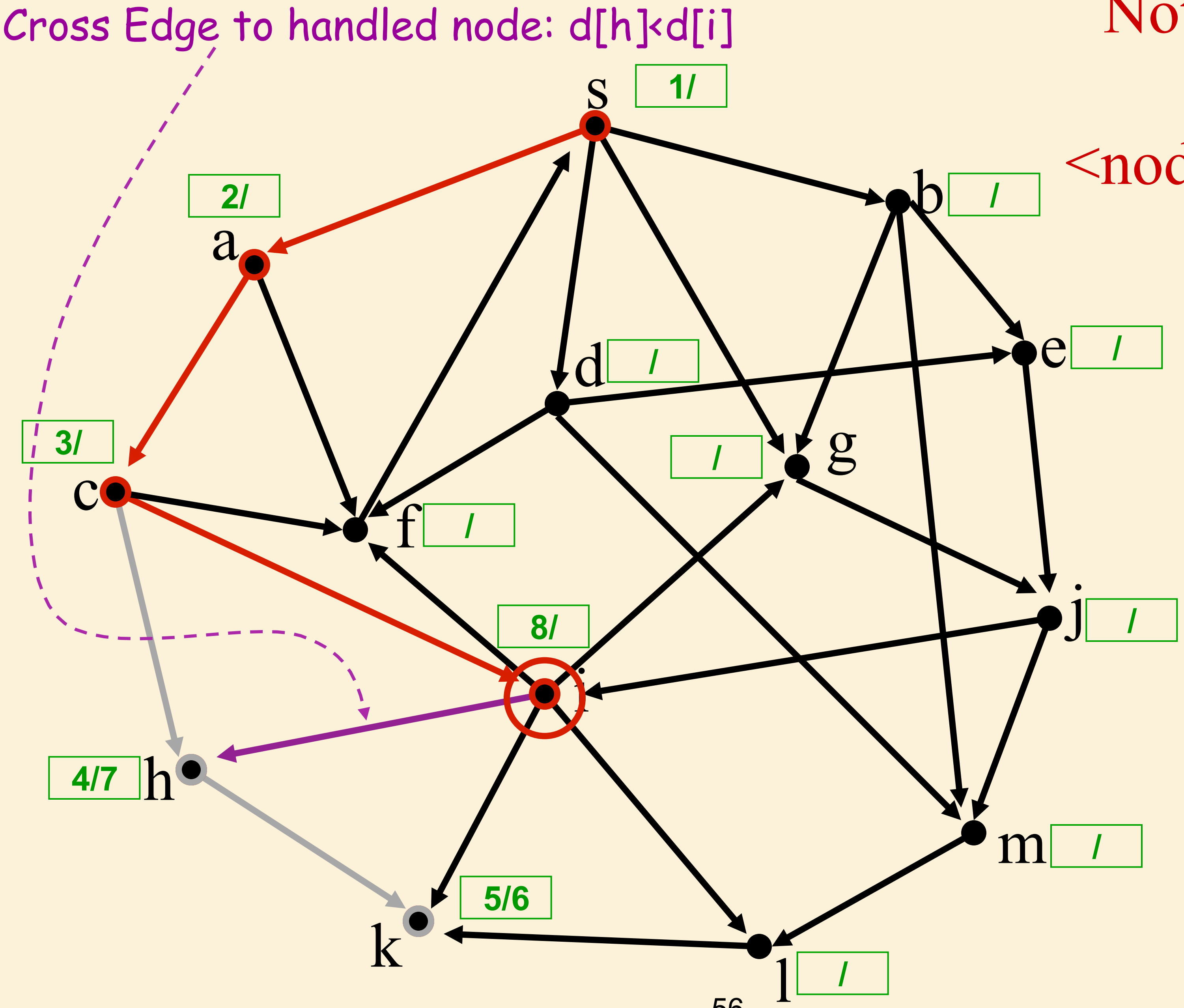
Stack
<node,# edges>

i,0
c,2
a,1
s,1

DFS

Found
Not Handled
Stack
<node,# edges>

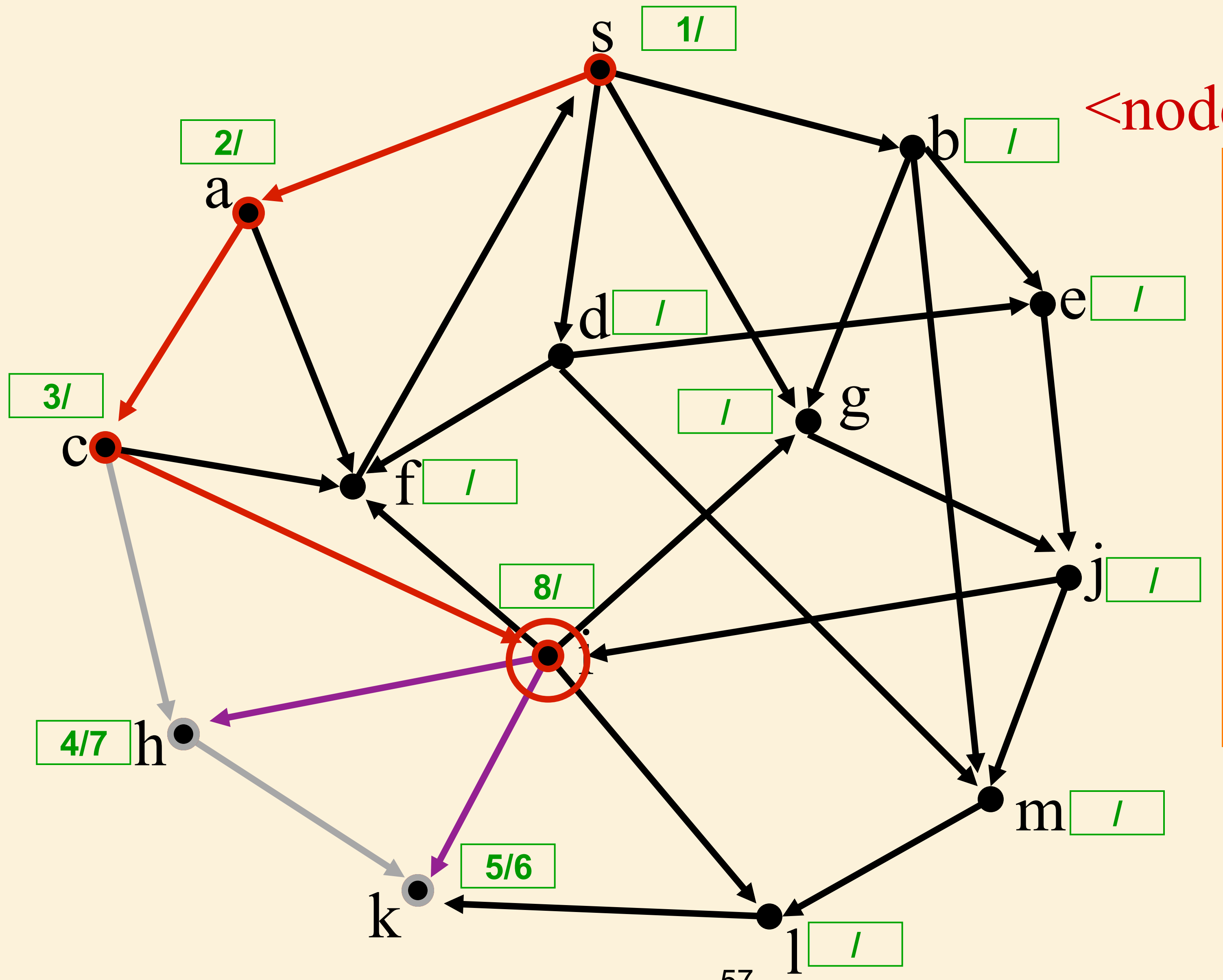


i,1
c,2
a,1
s,1

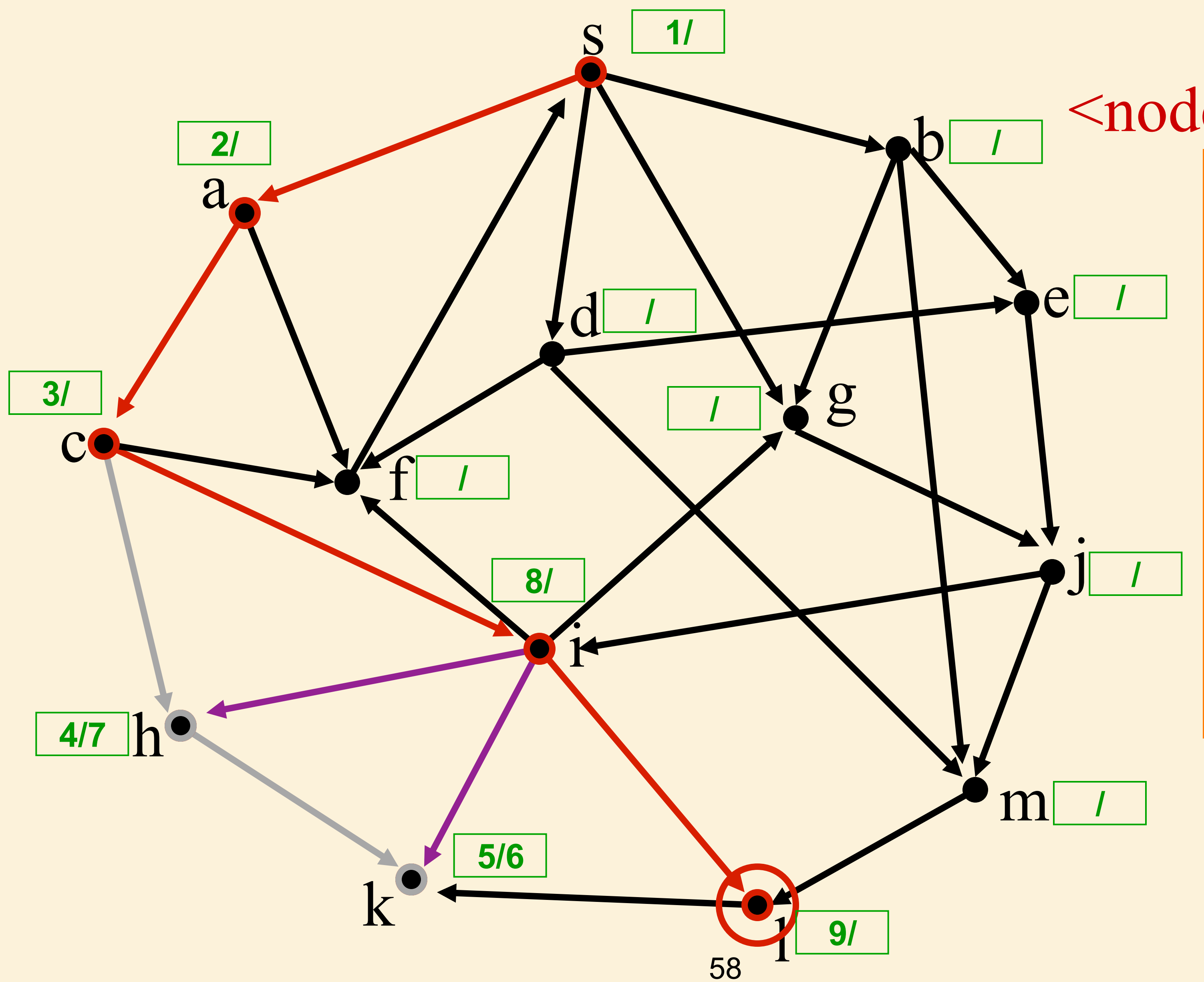
DFS

Found
Not Handled
Stack
<node,# edges>

i,2
c,2
a,1
s,1



DFS



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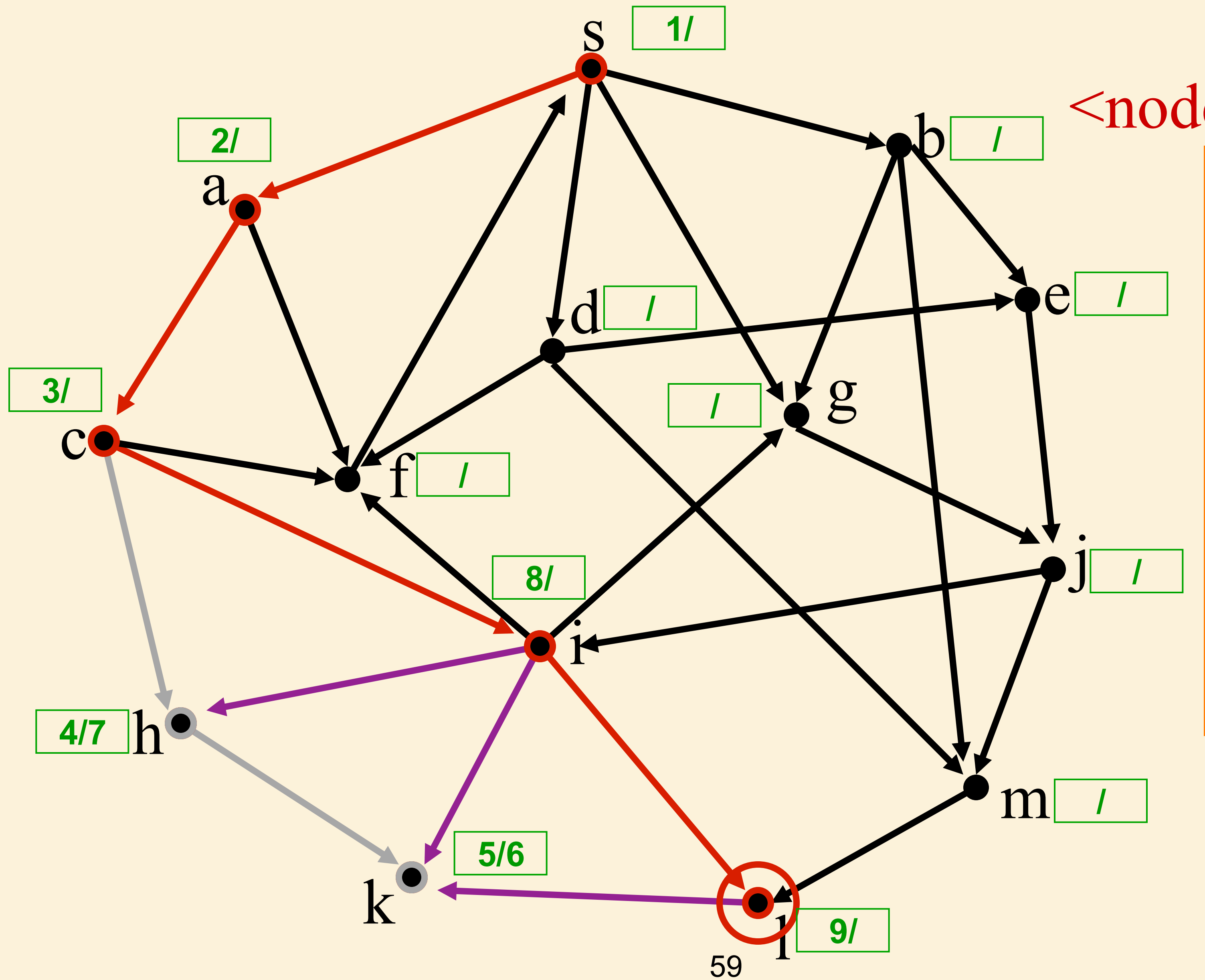
Not Handled
Stack
<node,# edges>

l,0
i,3
c,2
a,1
s,1

DFS



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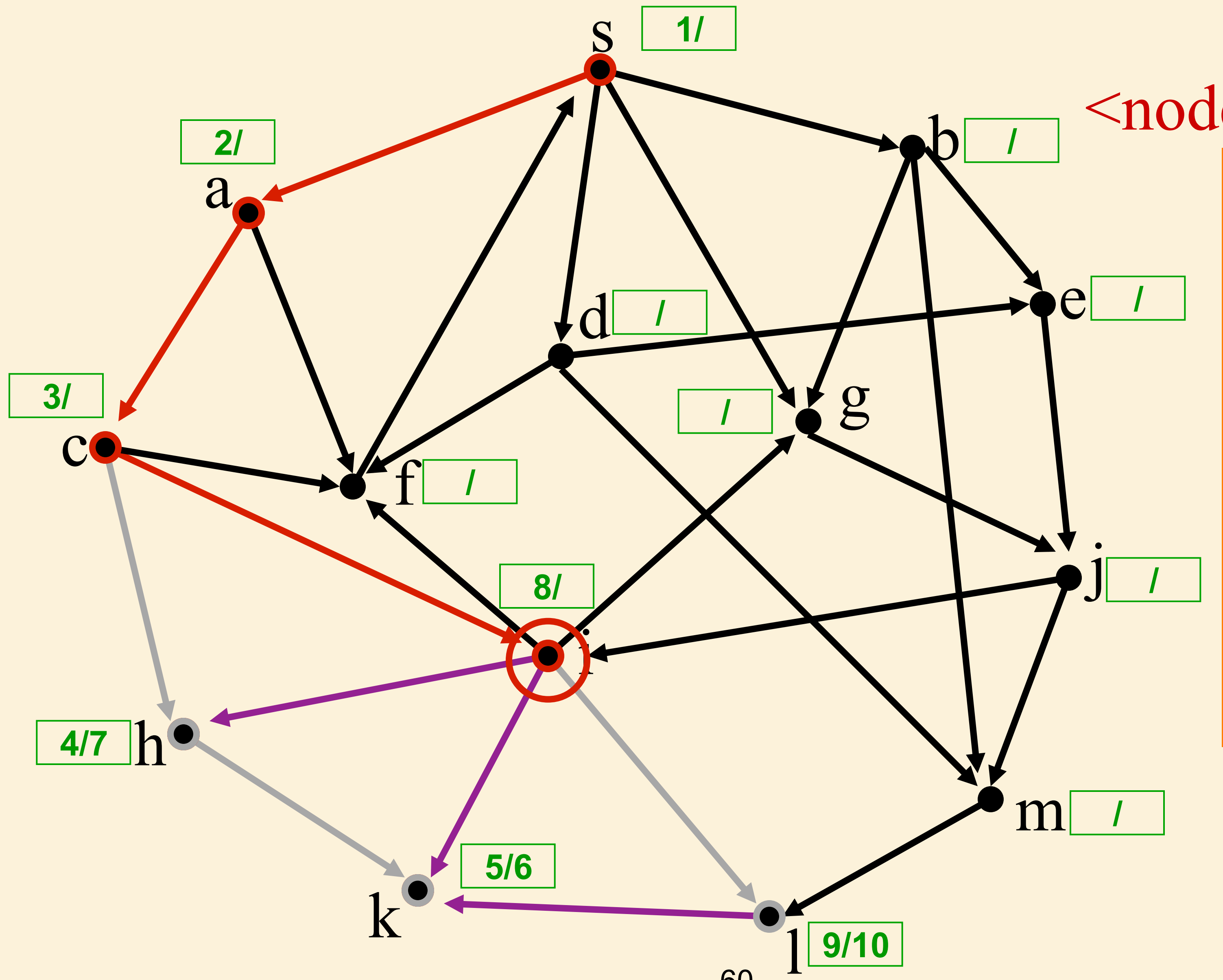
Stack
<node,# edges>

l,1
i,3
c,2
a,1
s,1

DFS



Found
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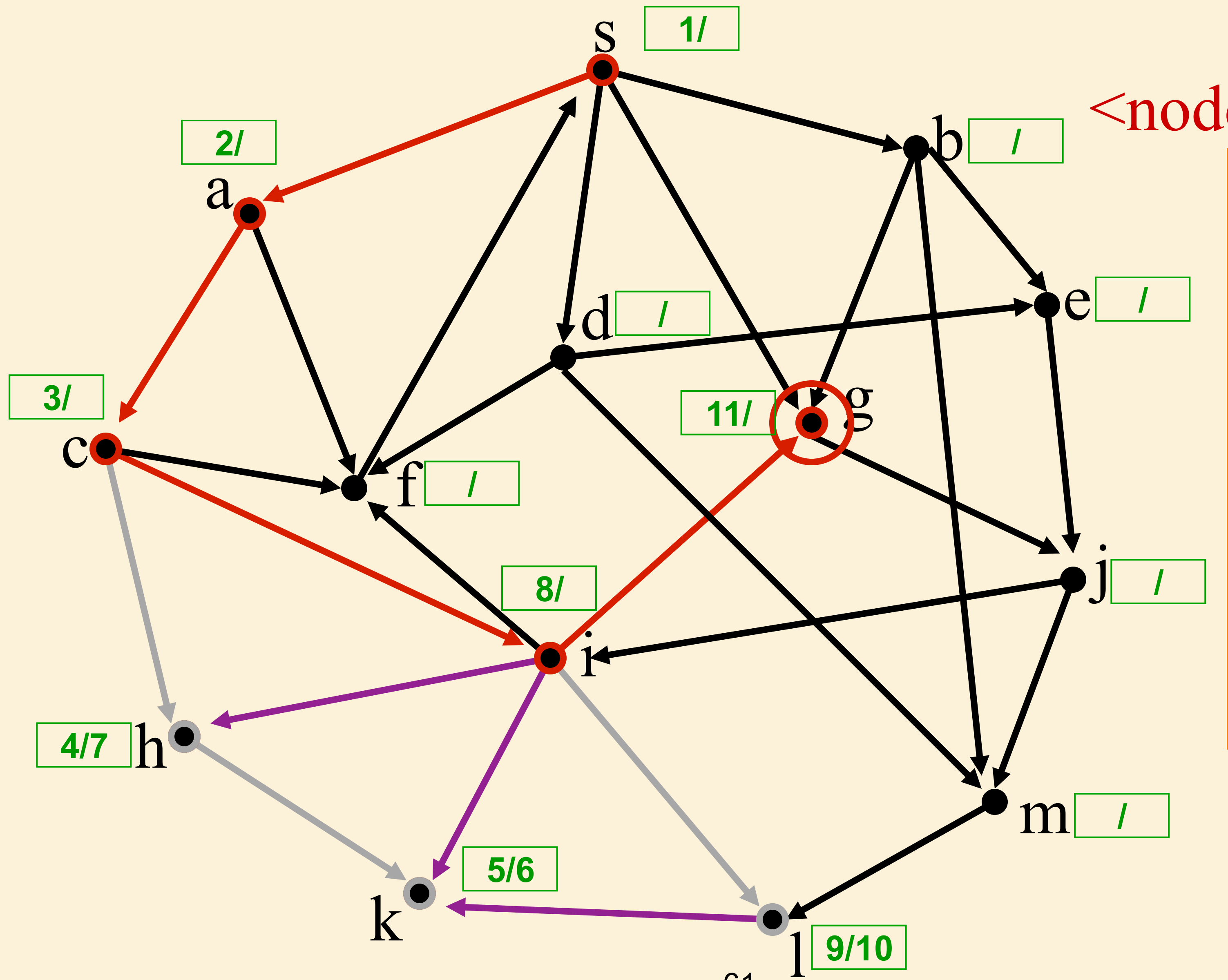
Stack
<node,# edges>

i,3
c,2
a,1
s,1

DFS

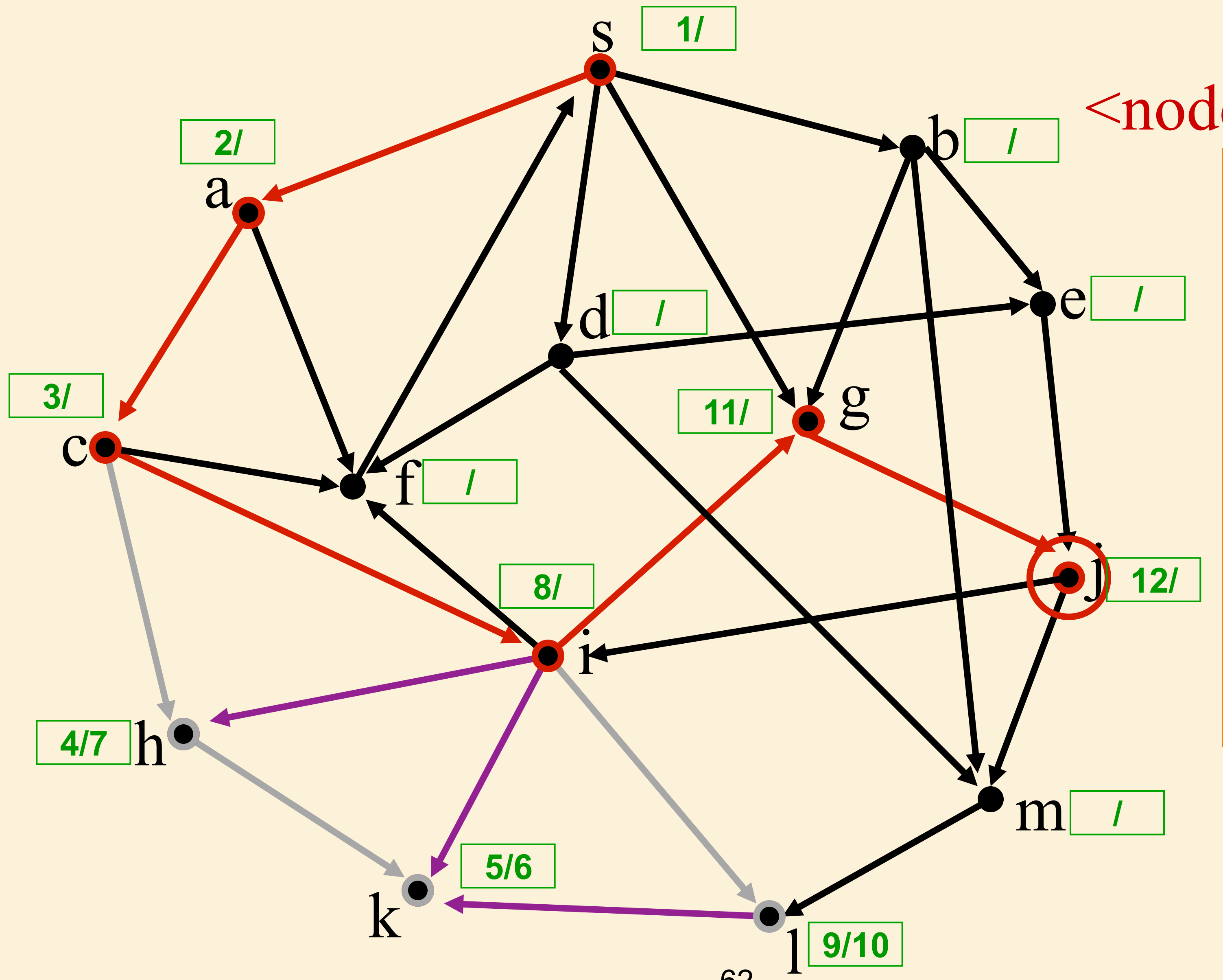


Found
Not Handled
Stack
<node,# edges>

g,0
i,4
c,2
a,1
s,1

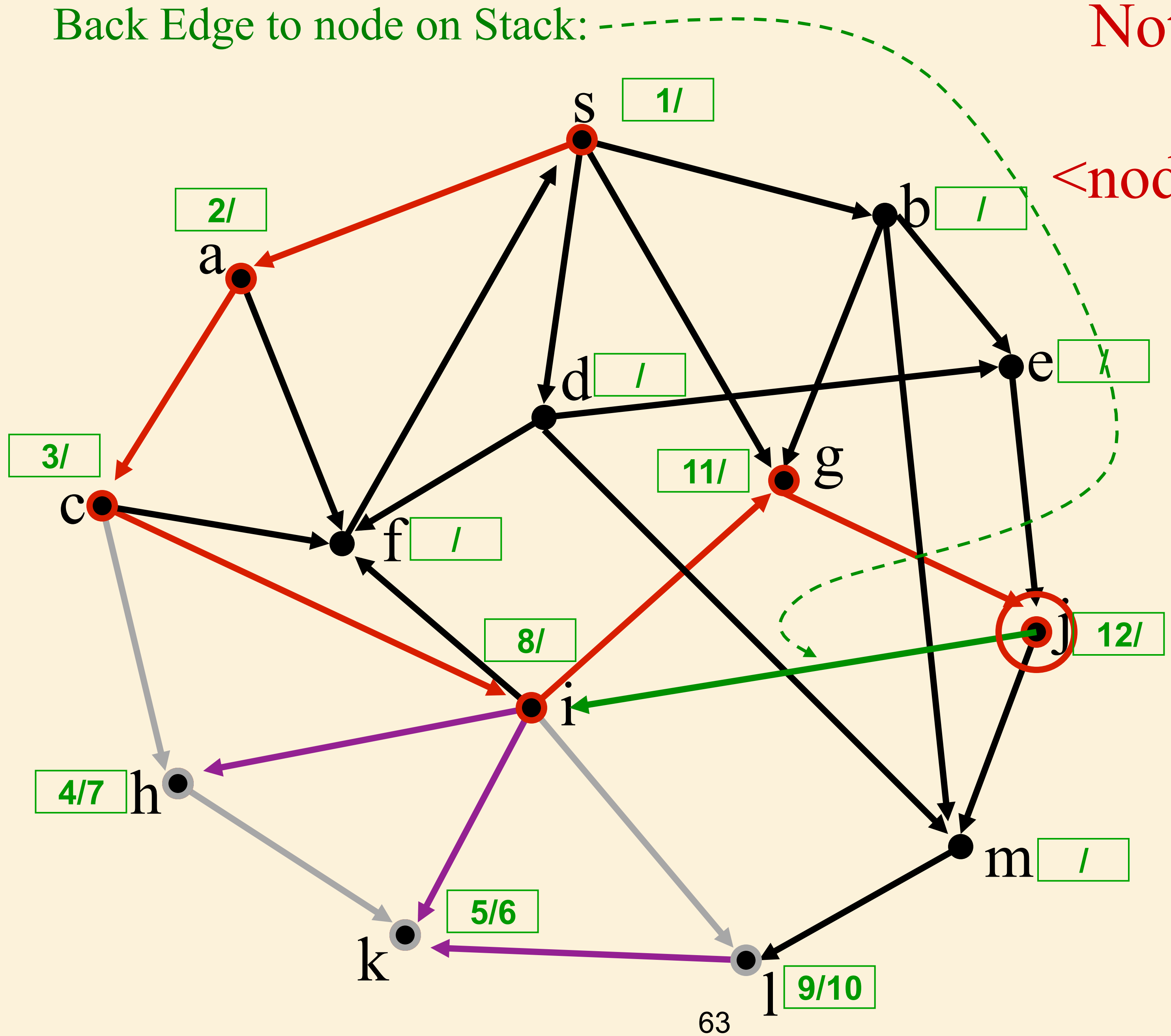
DFS

Found
Not Handled
Stack
<node,# edges>



j,0
g,1
i,4
c,2
a,1
s,1

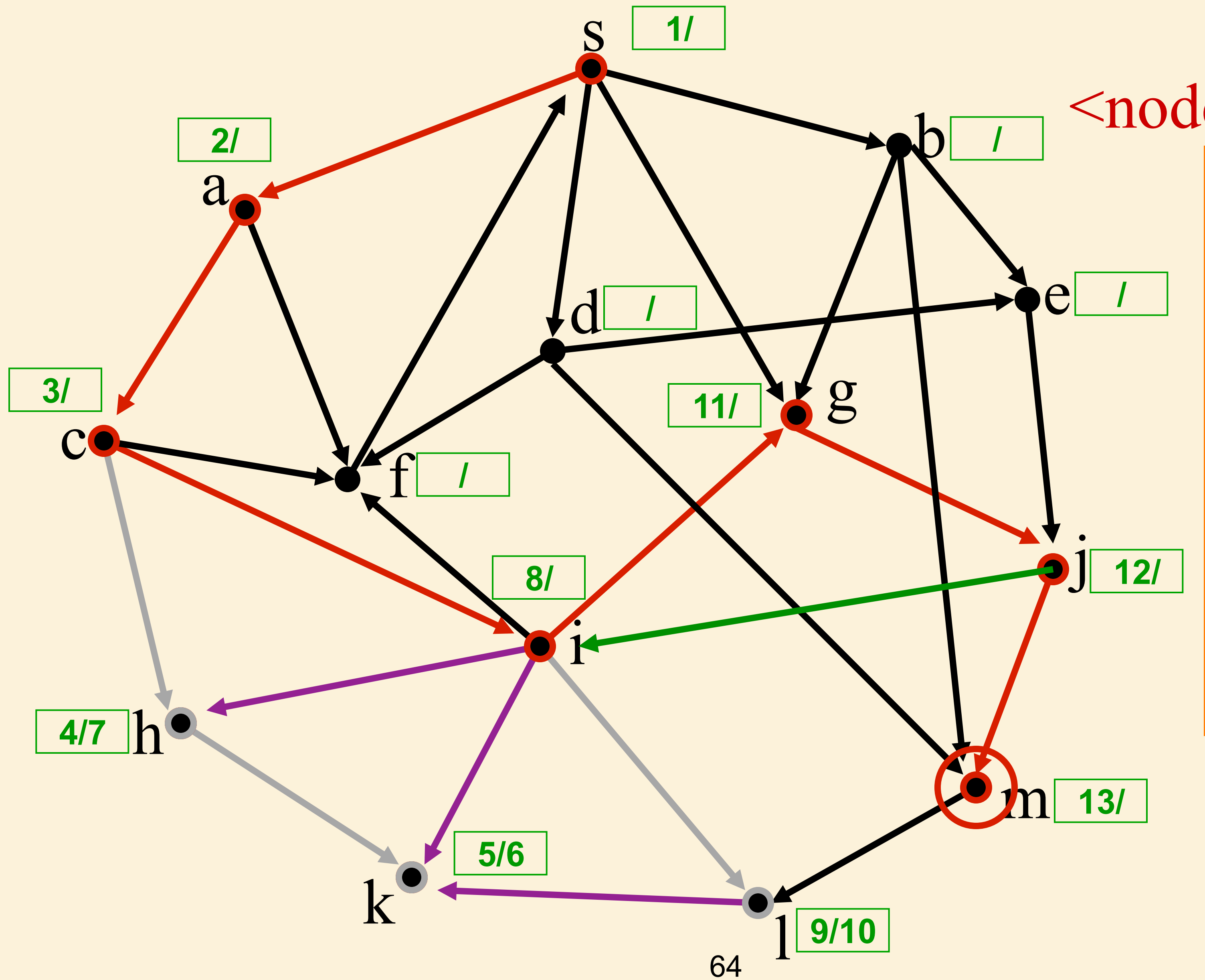
DFS



Found
Not Handled
Stack
<node,# edges>

j,1
g,1
i,4
c,2
a,1
s,1

DFS



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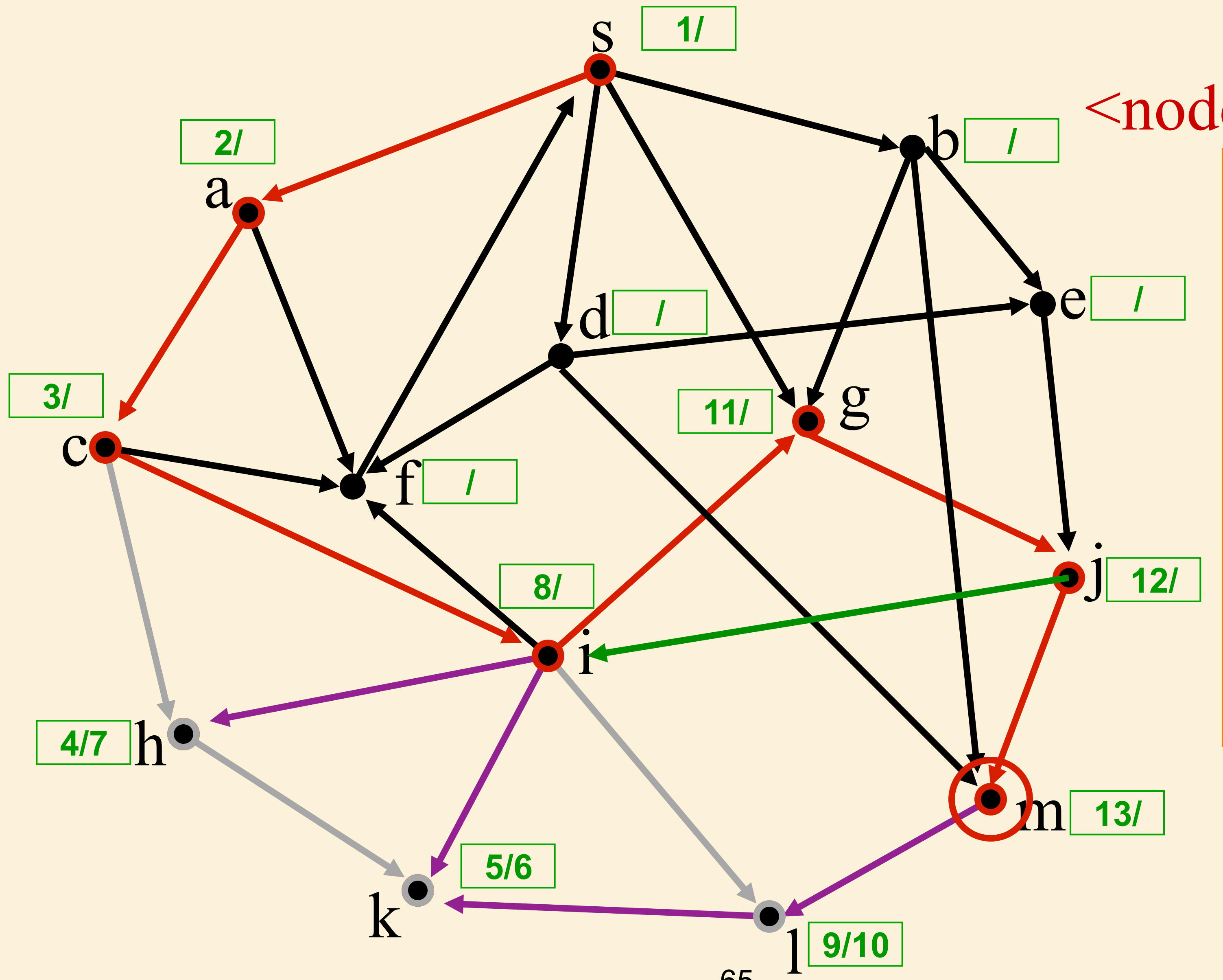
Not Handled
Stack
<node,# edges>

m,0
j,2
g,1
i,4
c,2
a,1
s,1

DFS



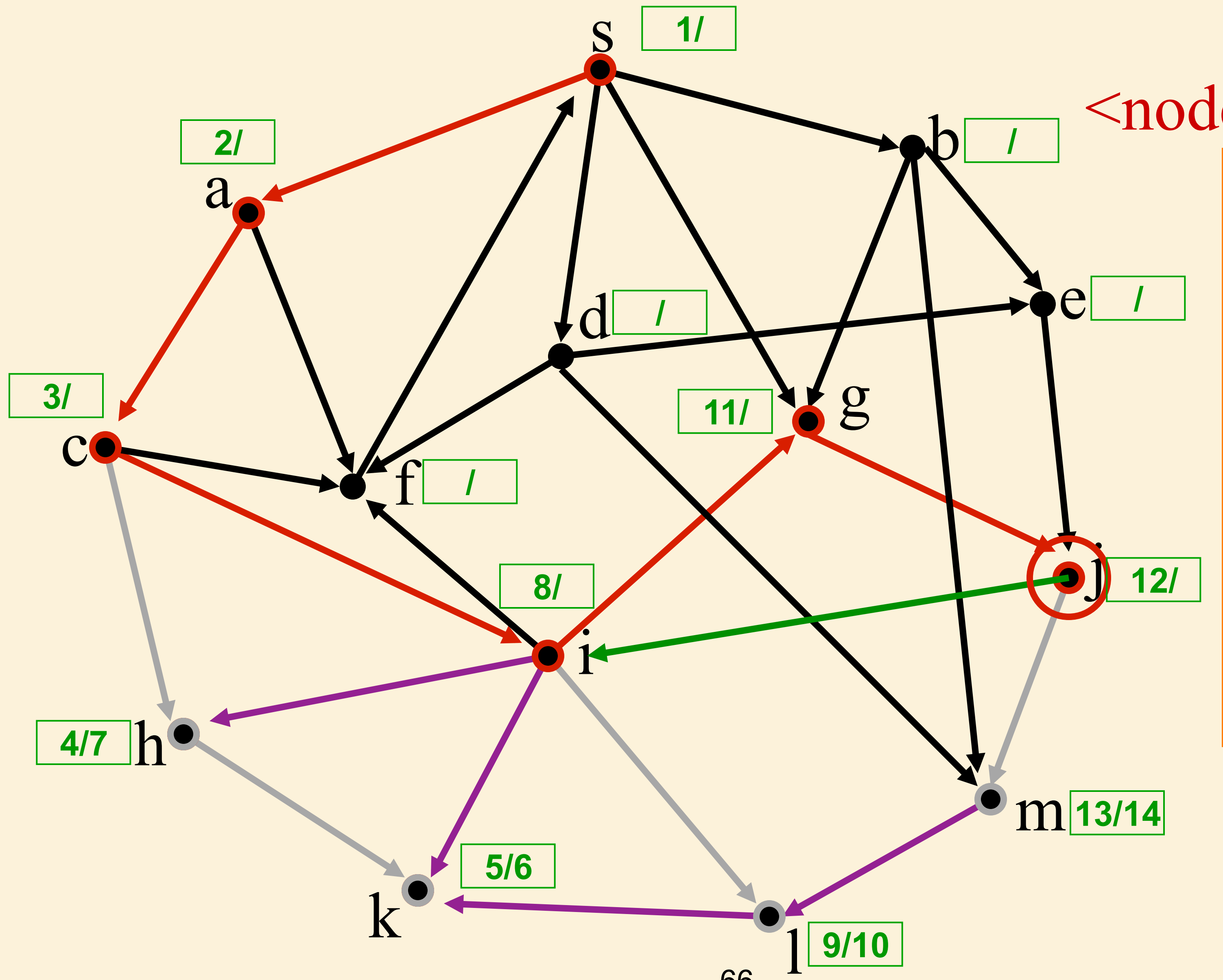
Found
Not Handled
Stack
<node,# edges>

m,1
j,2
g,1
i,4
c,2
a,1
s,1

DFS

Found
Not Handled
Stack
<node,# edges>

j,2
g,1
i,4
c,2
a,1
s,1



DFS

Found
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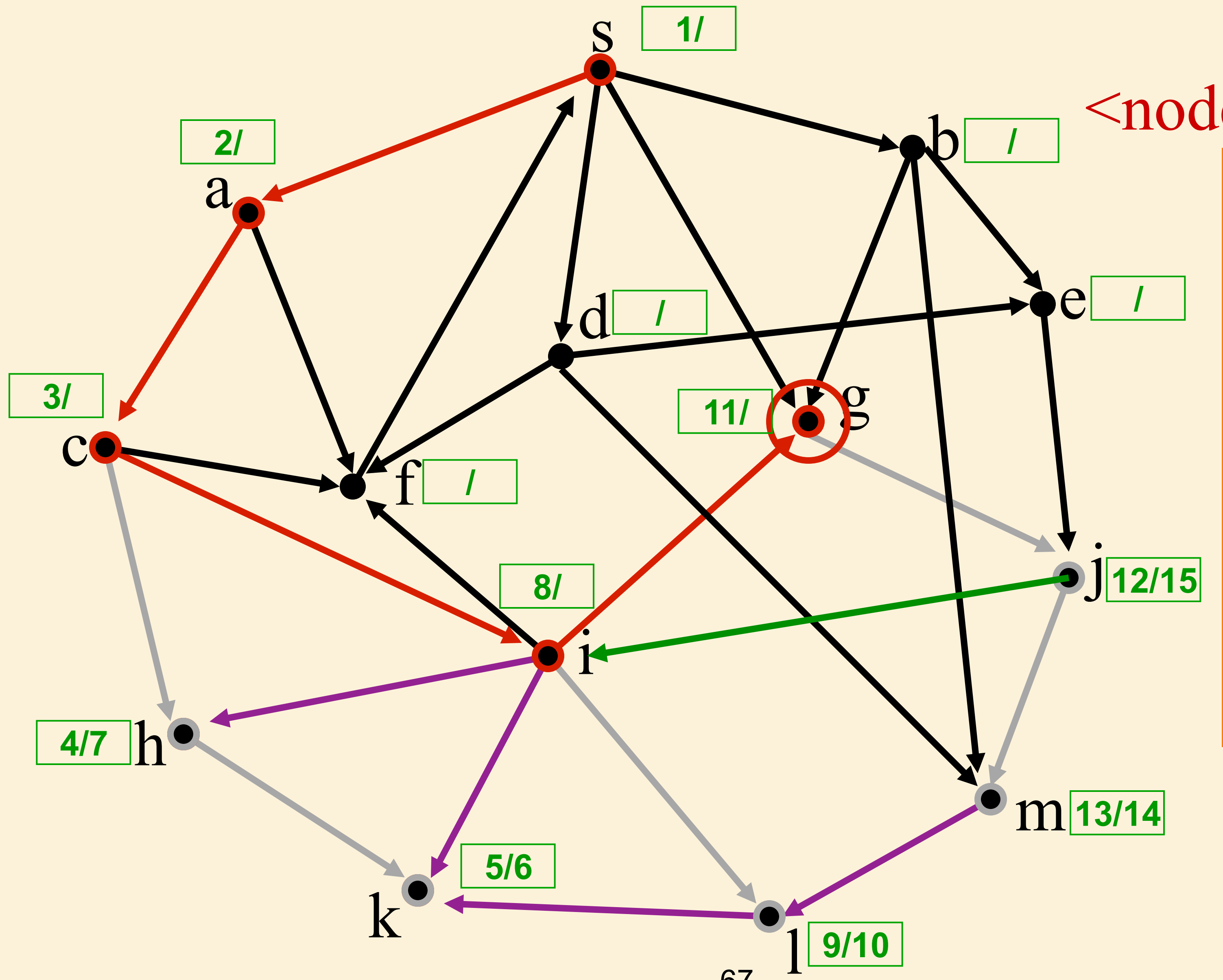
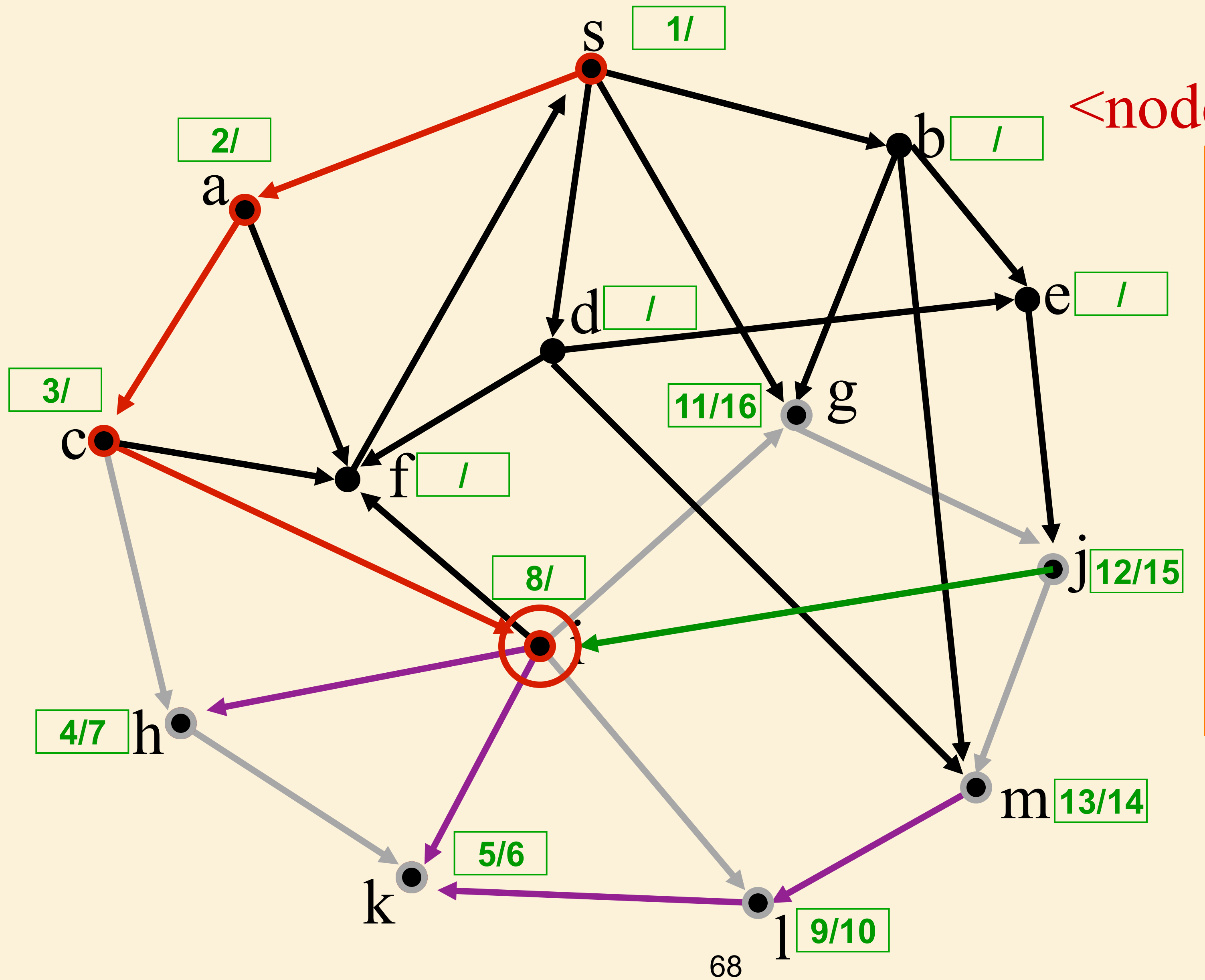
Stack
<node,# edges>

g,1
i,4
c,2
a,1
s,1



DFS



Found
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Stack
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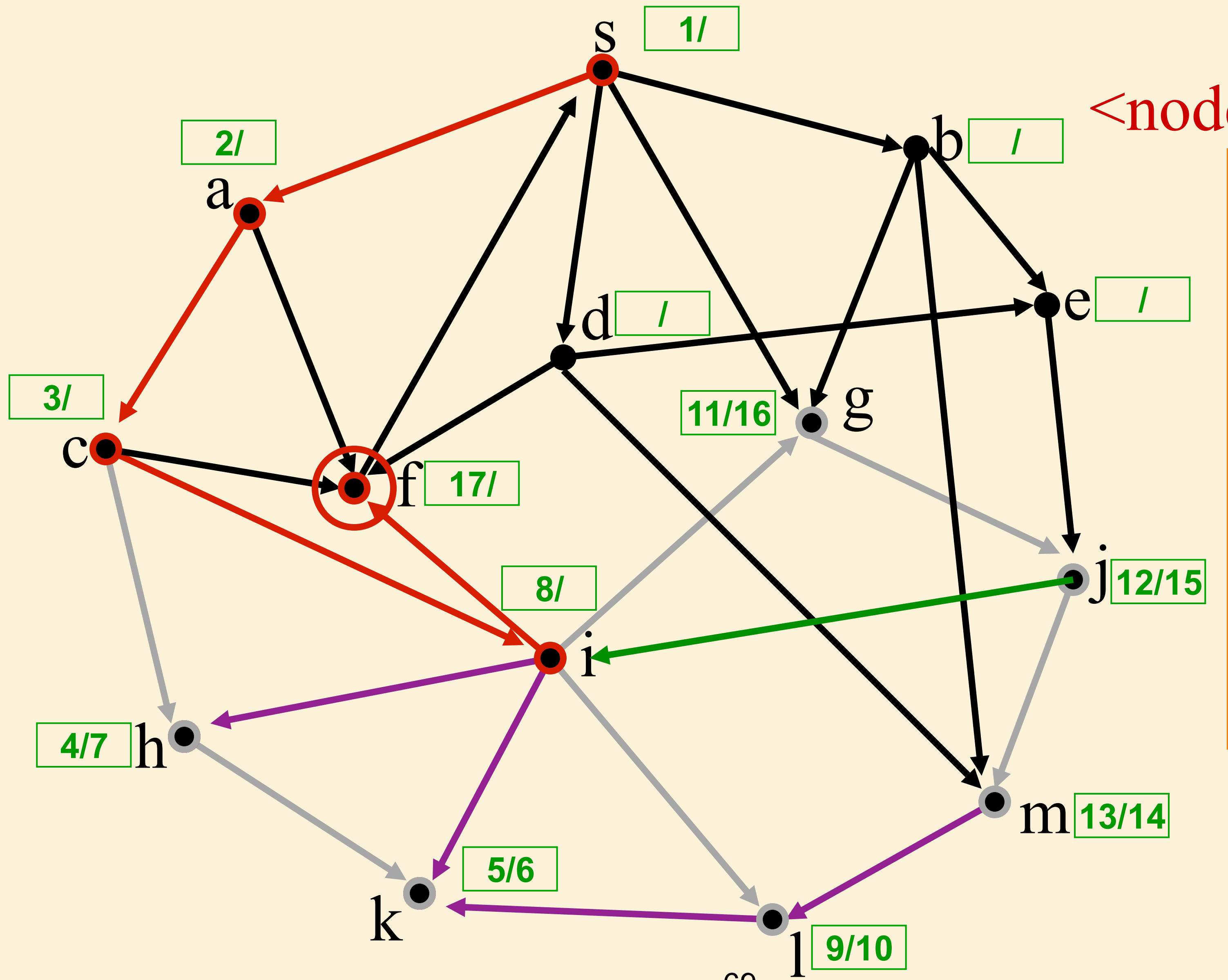
i,4
c,2
a,1
s,1

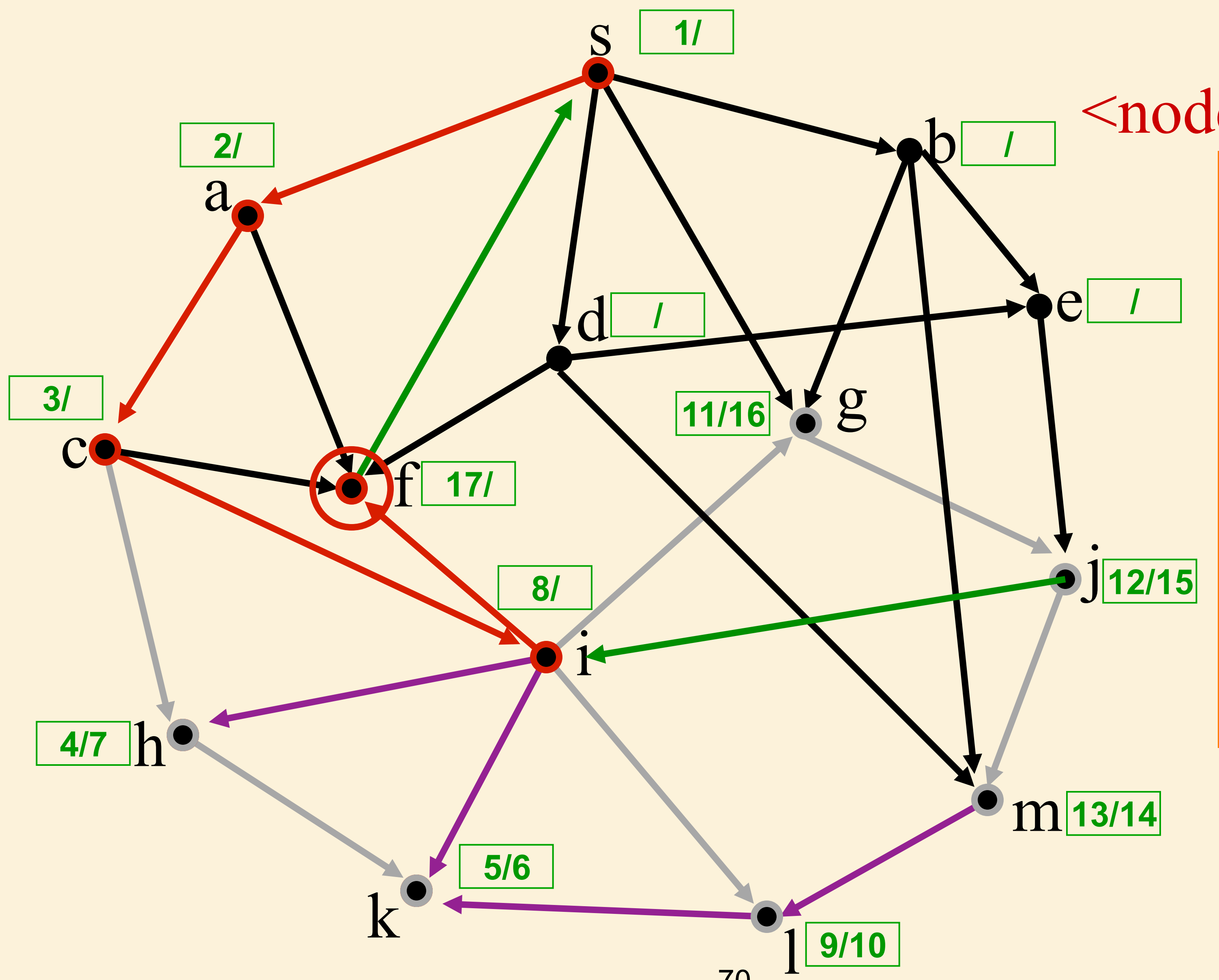
DFS

Found
Not Handled
Stack
<node,# edges>



f,0
i,5
c,2
a,1
s,1

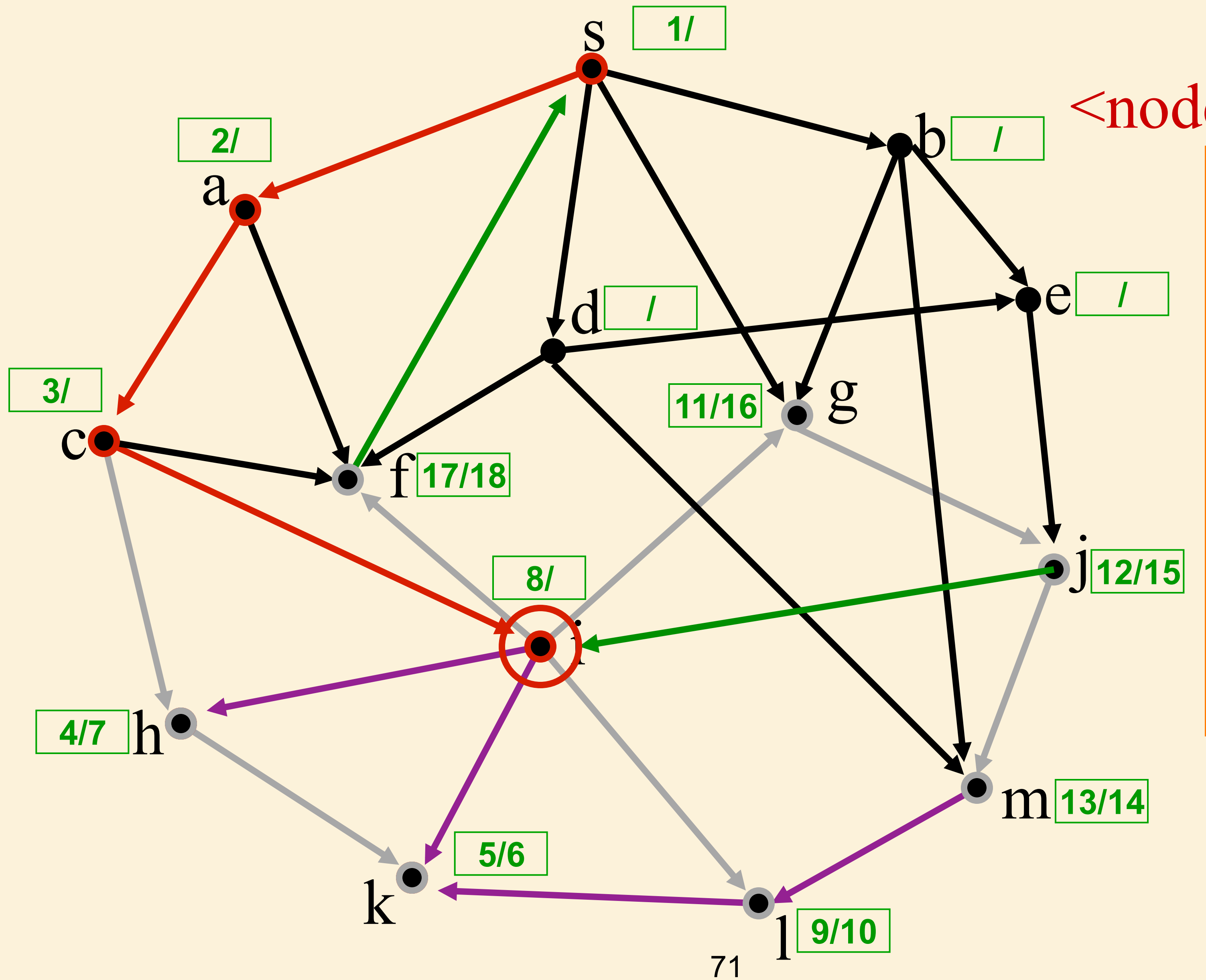
DFS



Found
Not Handled
Stack
<node,# edges>

f,1
i,5
c,2
a,1
s,1

DFS



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Stack
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i,5
c,2
a,1
s,1

DFS

Found
Not Handled
Stack
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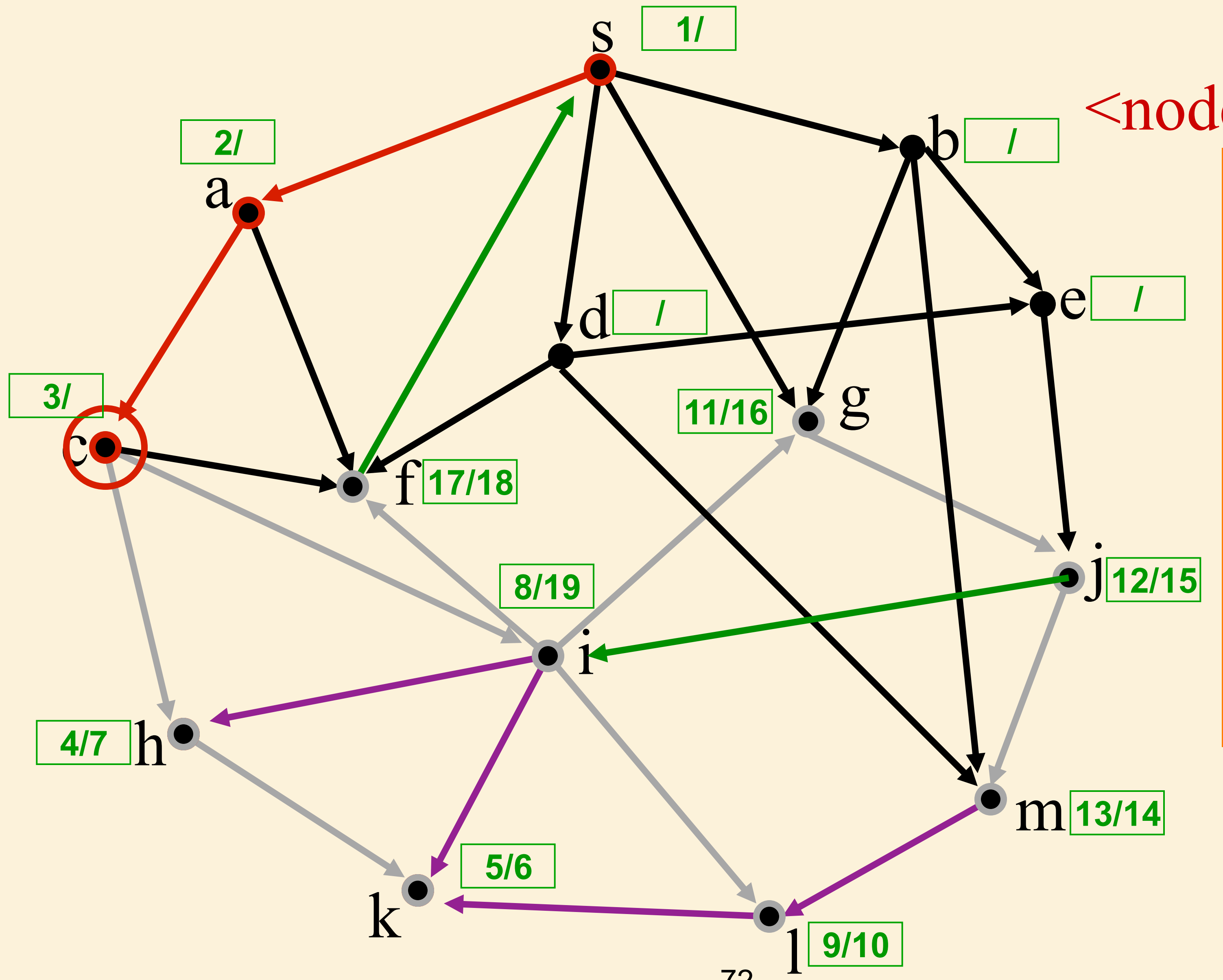
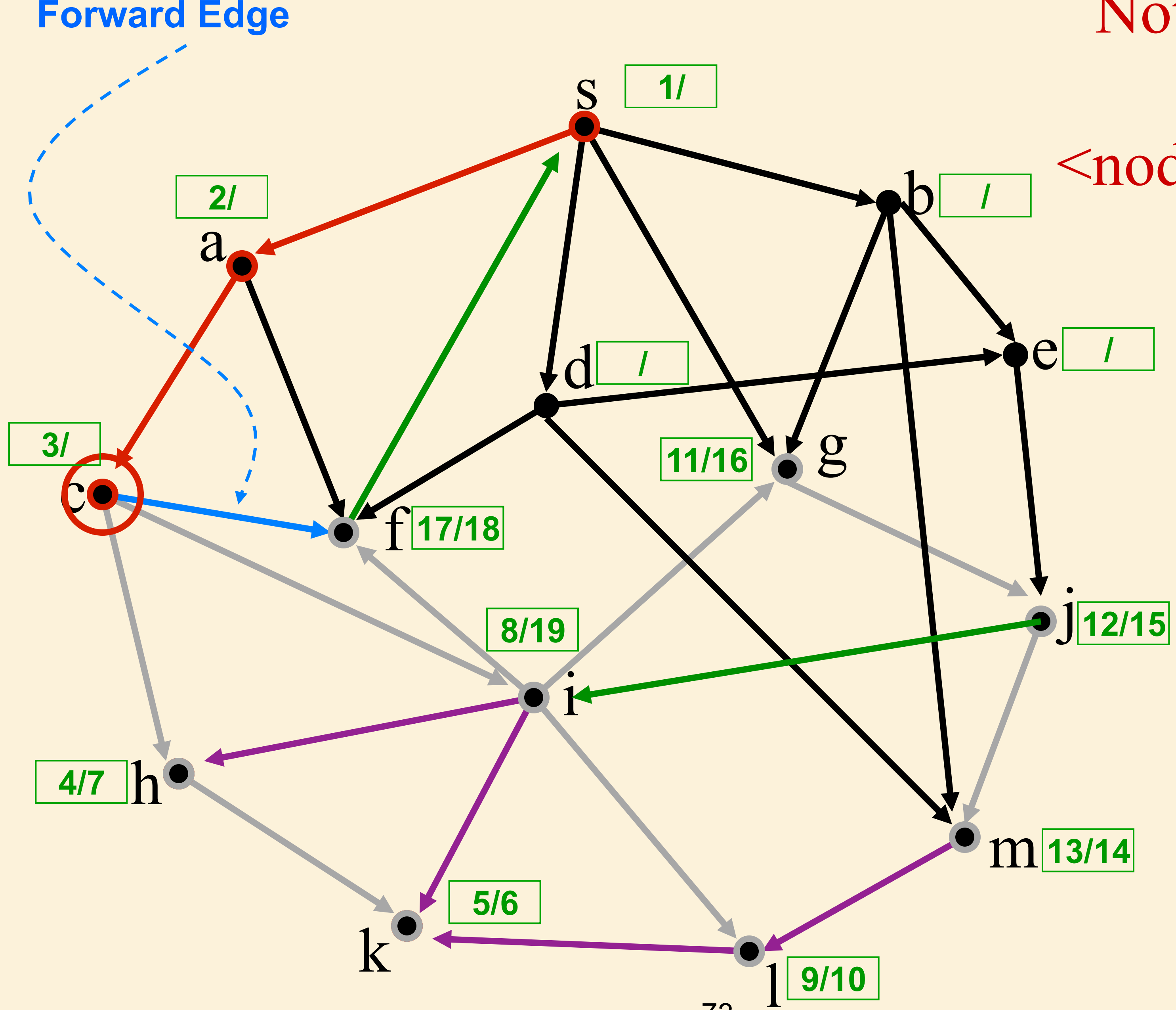


c,2
a,1
s,1

DFS

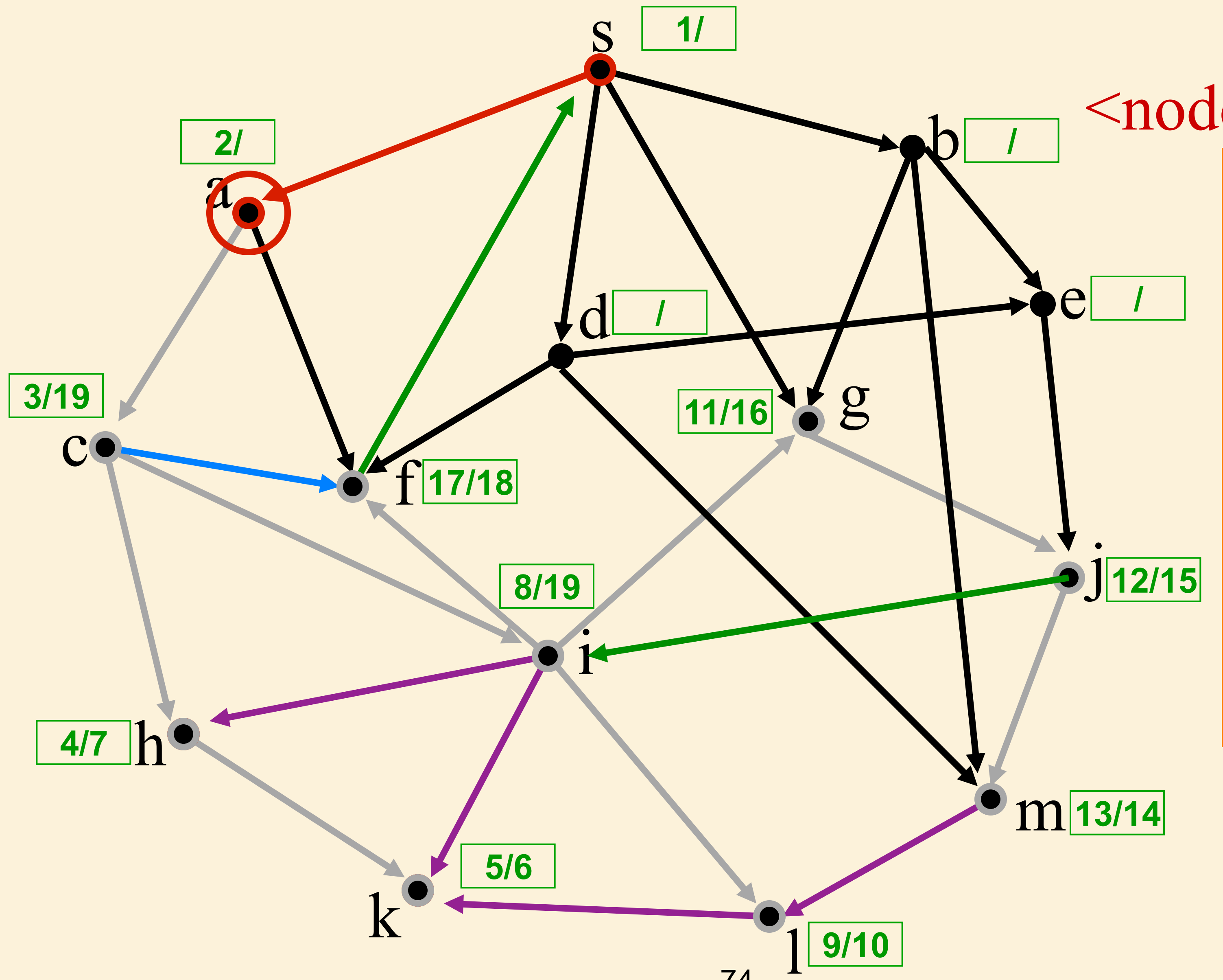


Found
Not Handled
Stack
<node,# edges>

c,3
a,1
s,1

DFS

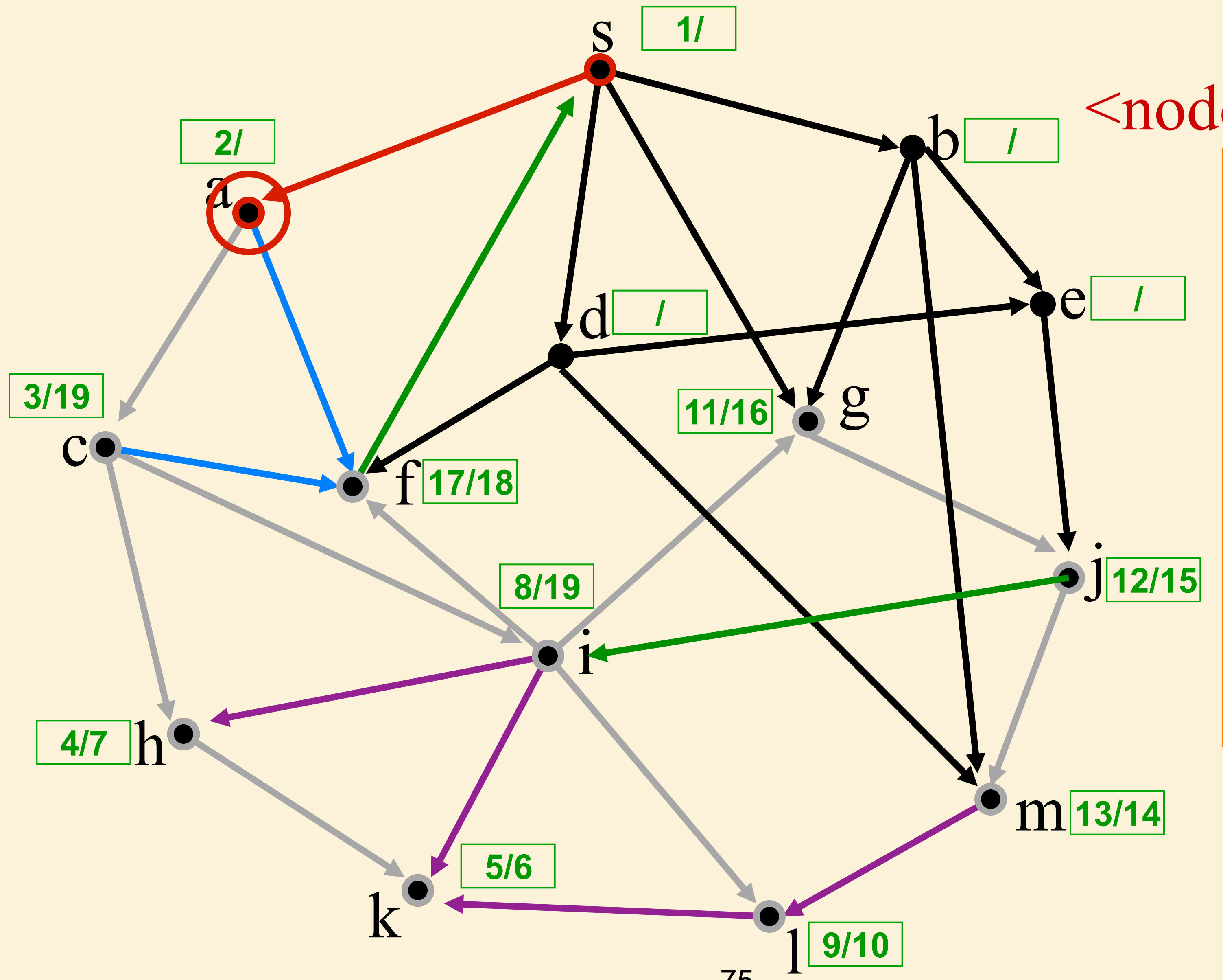
Found
Not Handled
Stack
<node,# edges>



a,1
s,1

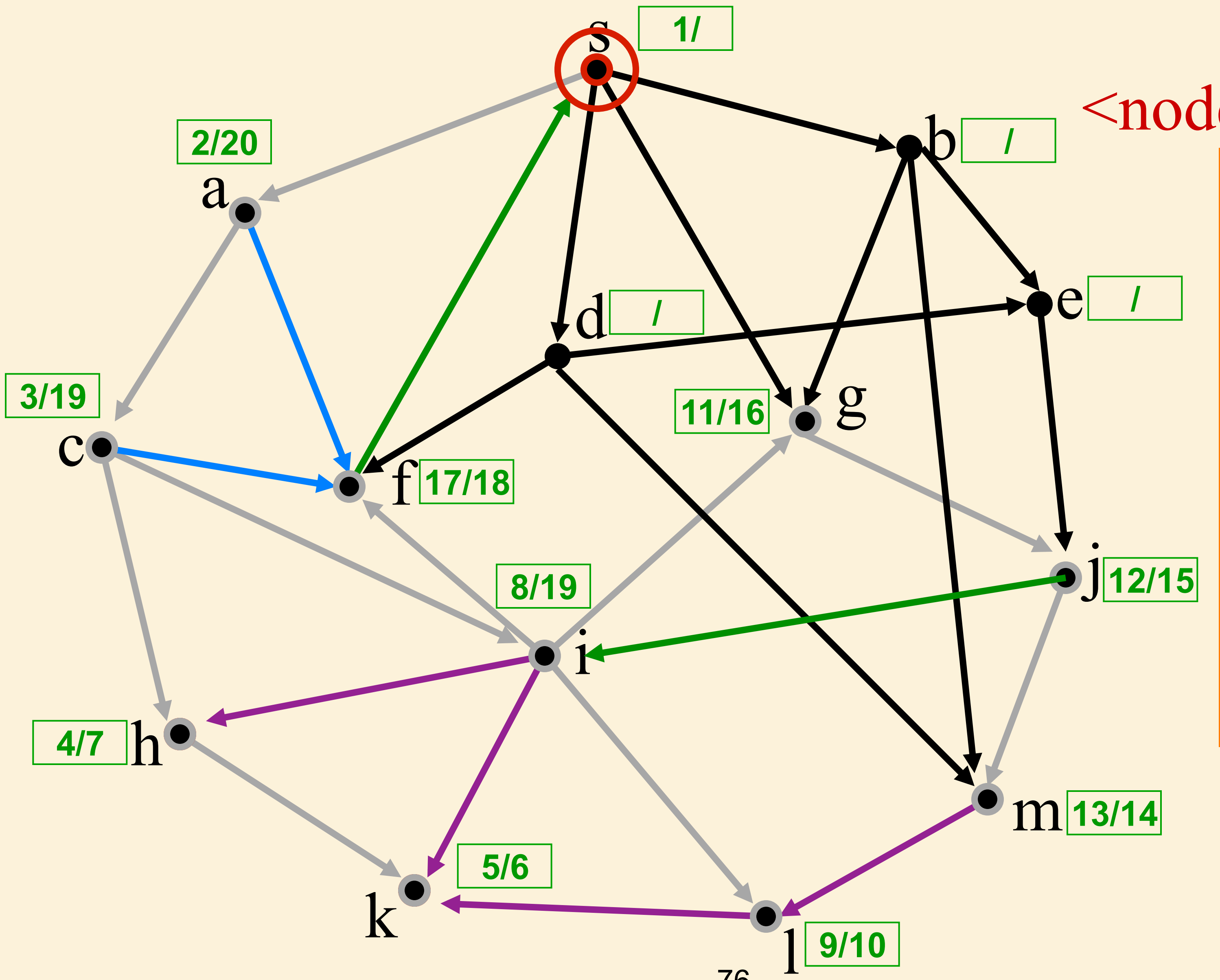
DFS

Found
Not Handled
Stack
<node,# edges>



a,2
s,1

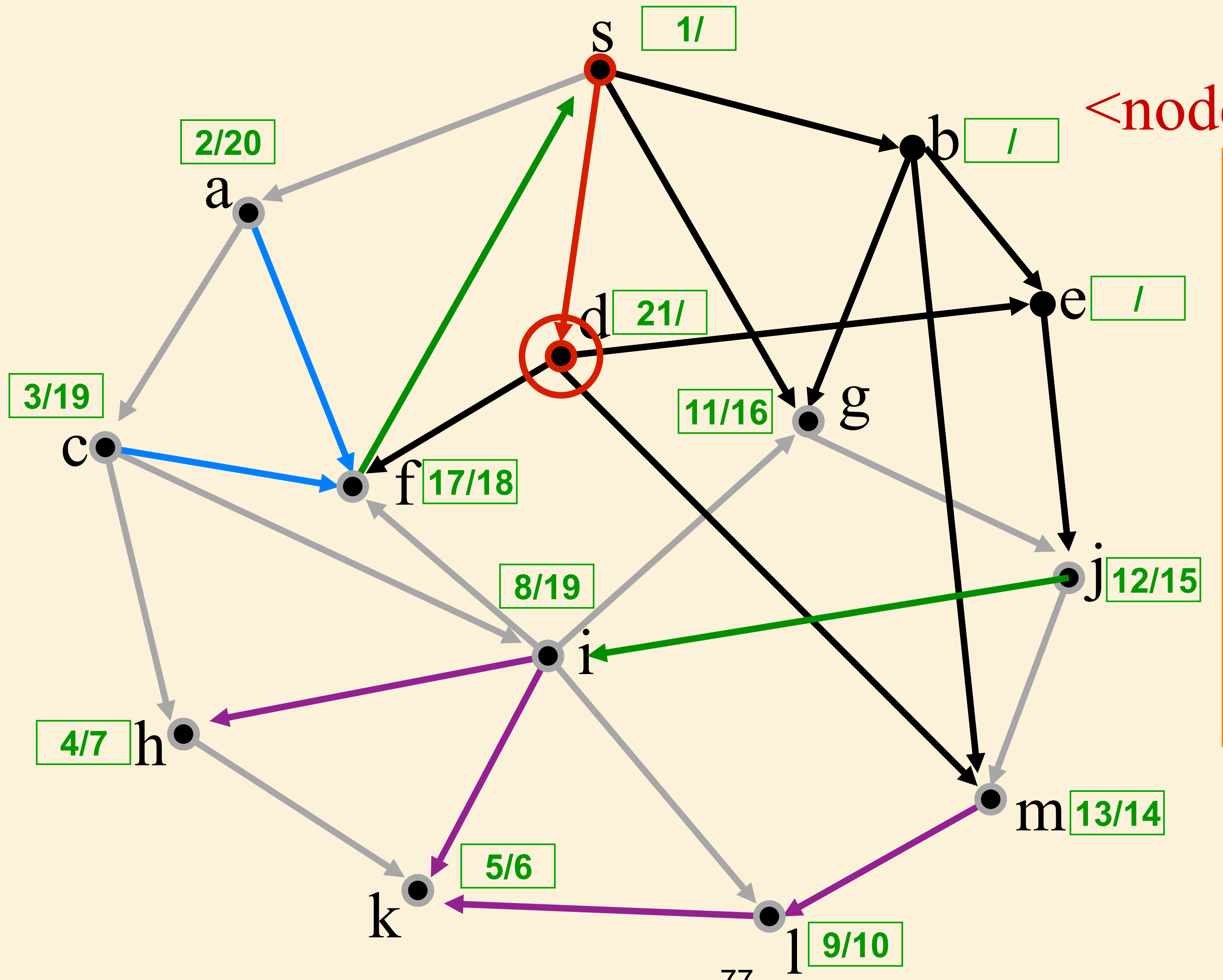
DFS



Found
Not Handled
Stack
<node,# edges>

s,1

DFS



Found
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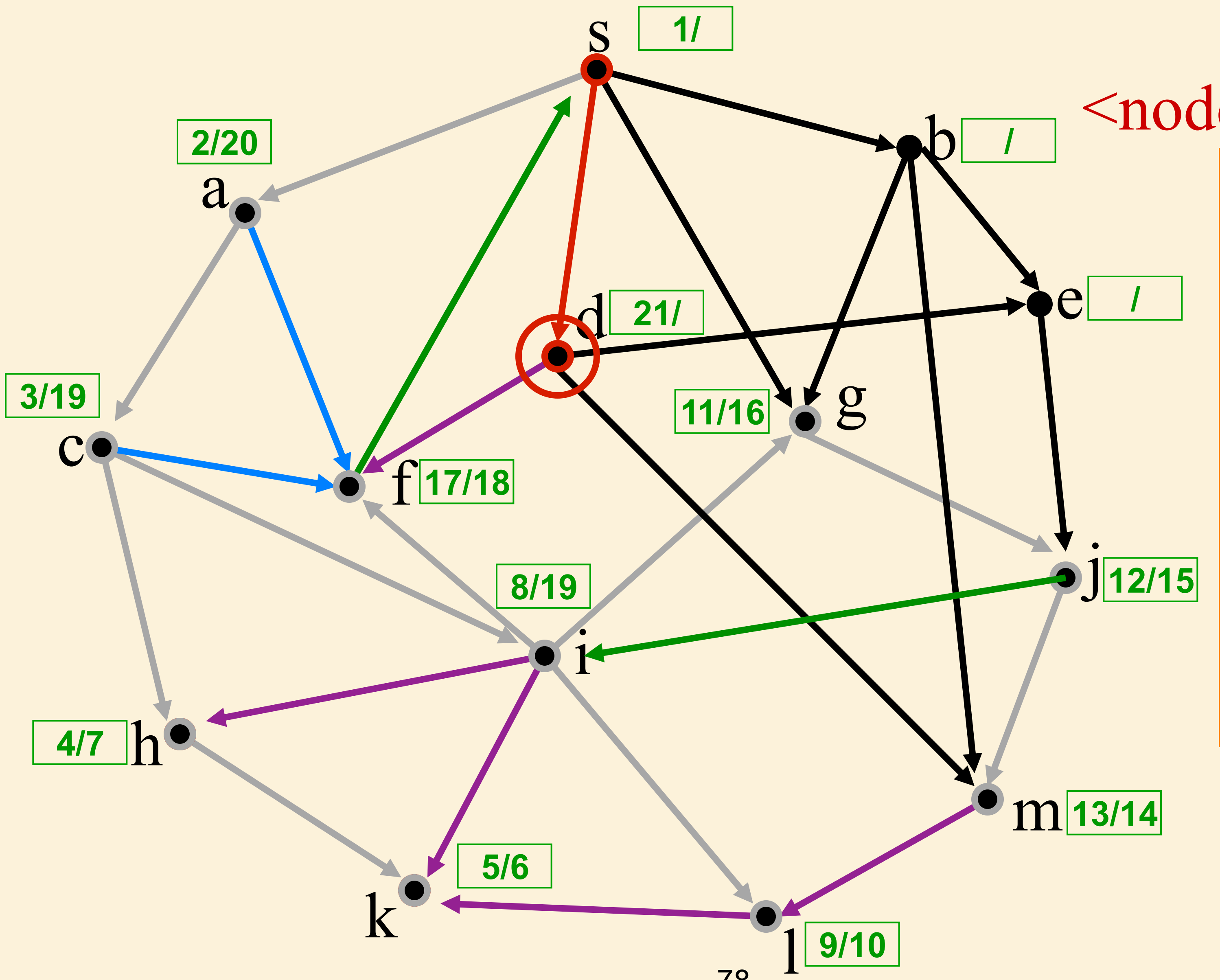
Stack
<node,# edges>

d,0
s,2

DFS



Found
Not Handled
Stack
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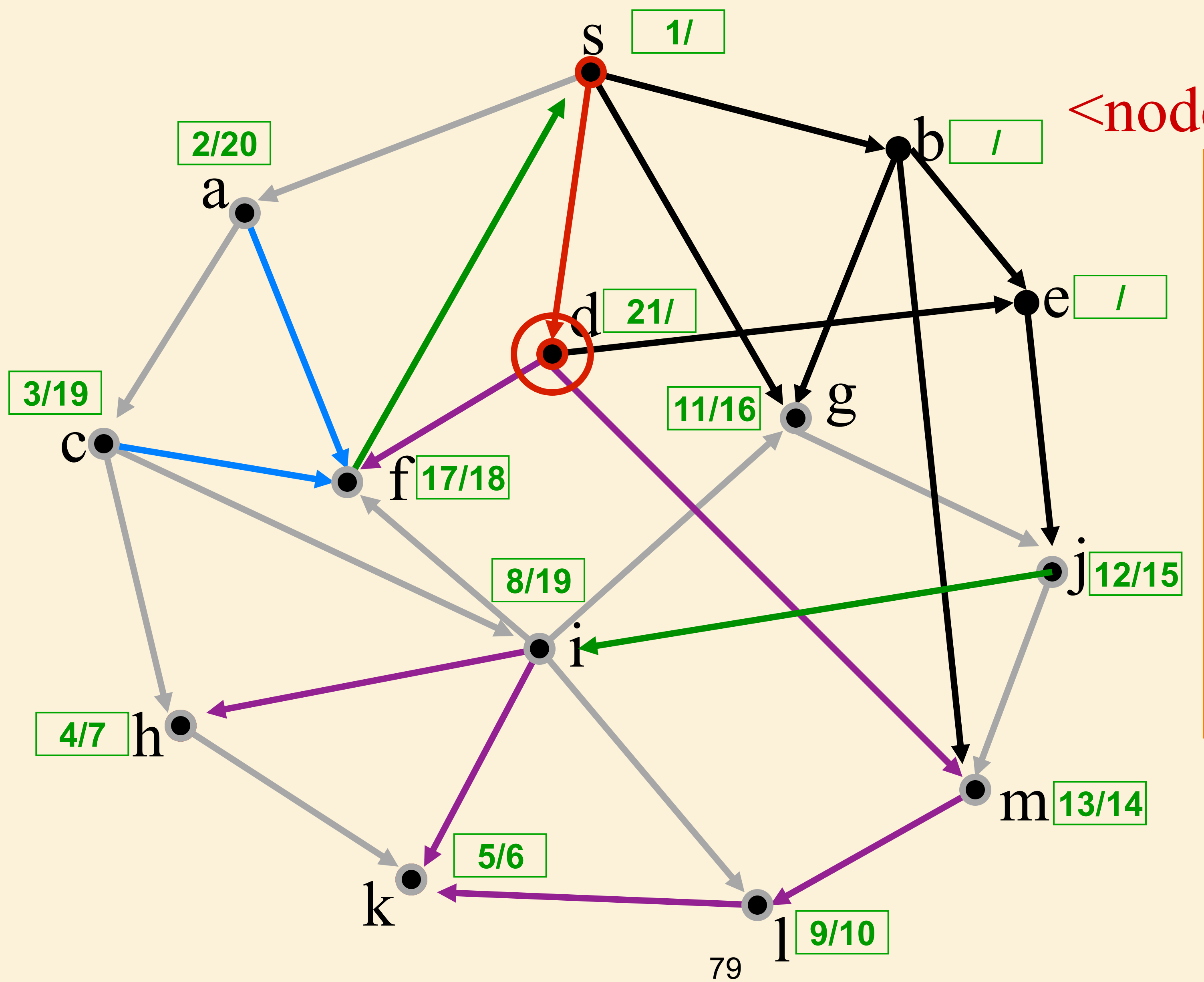
d,1
s,2

DFS



Found
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Stack
<node,# edges>

d,2
s,2

DFS

Found
Not Handled
Stack
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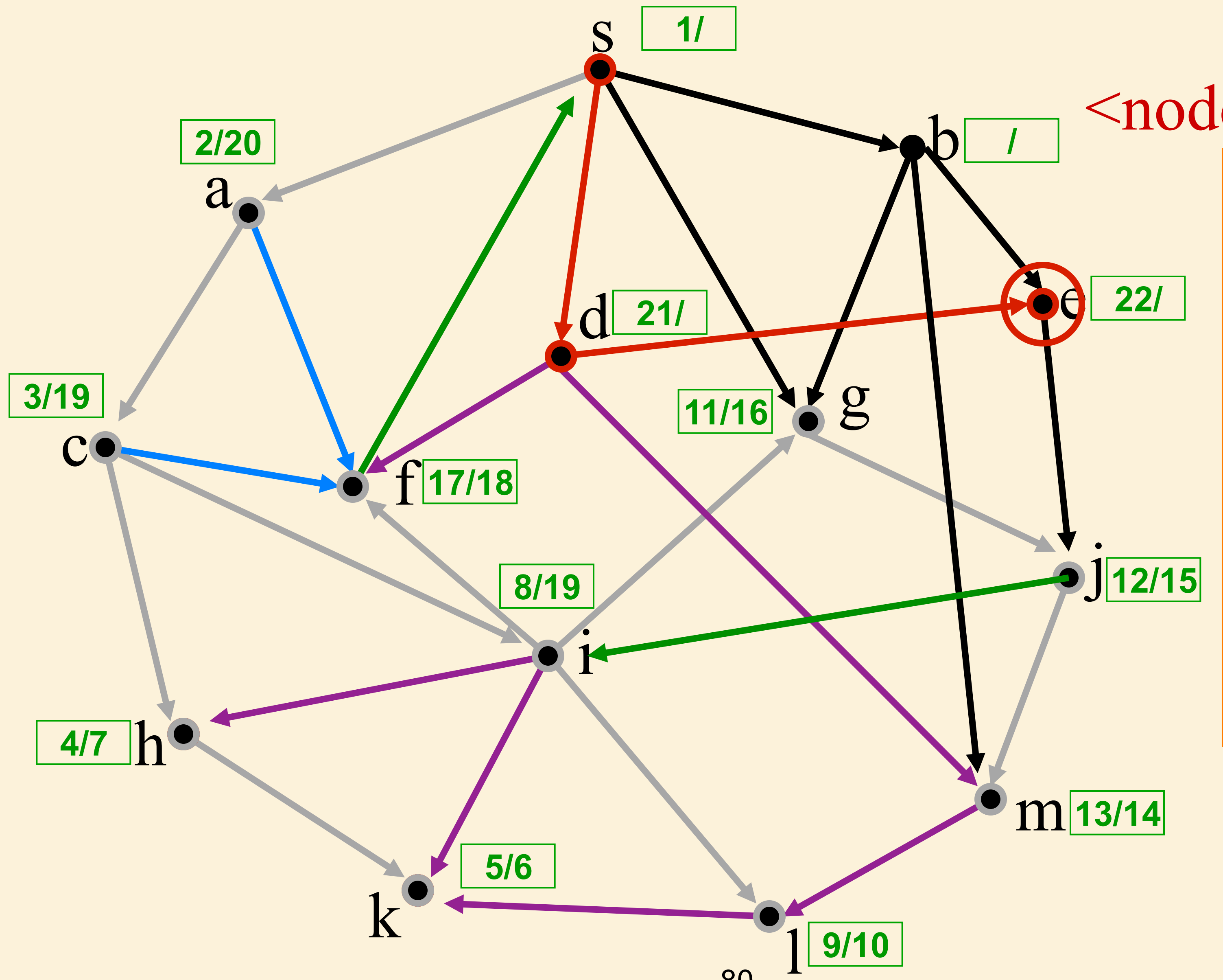
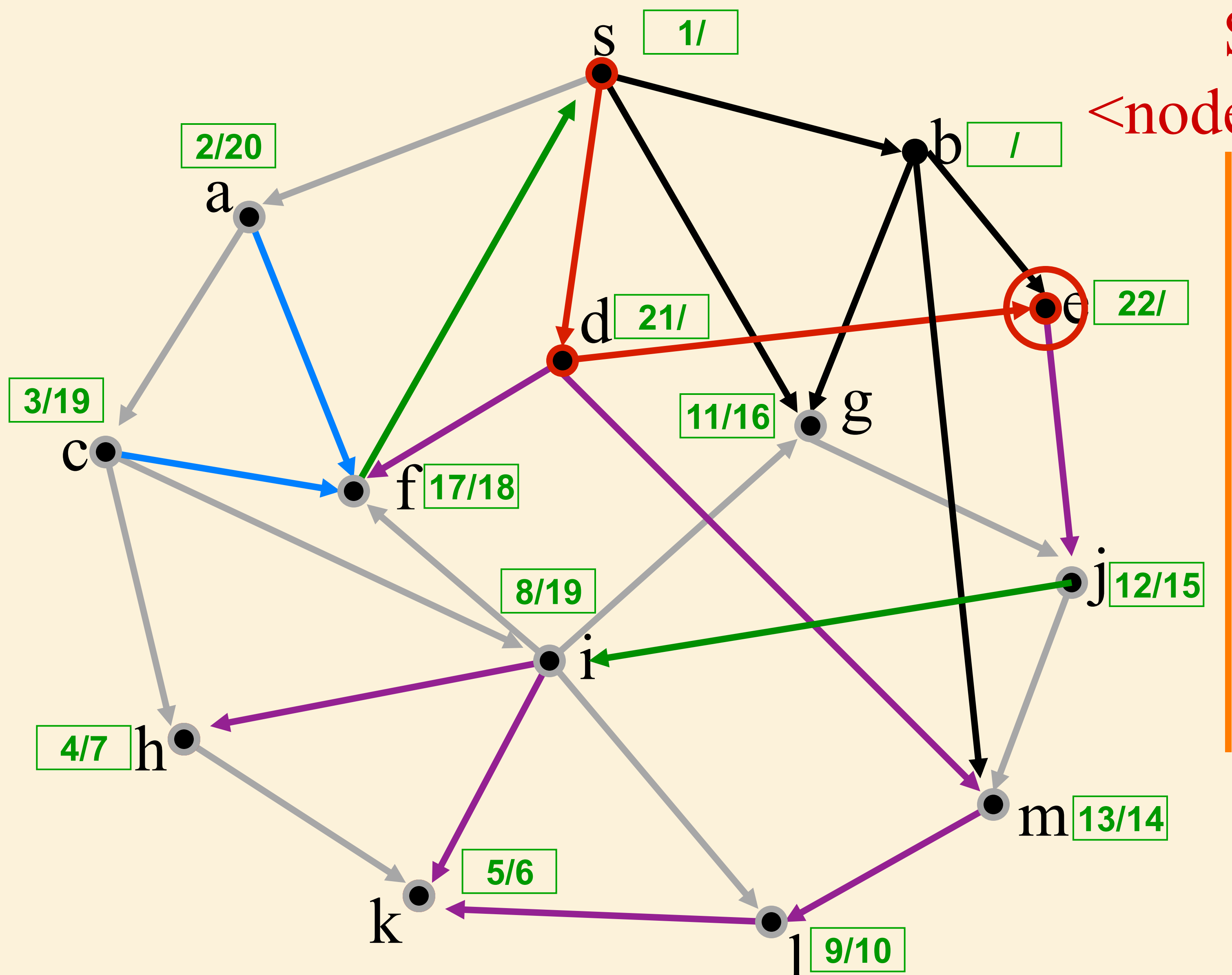


e,0
d,3
s,2

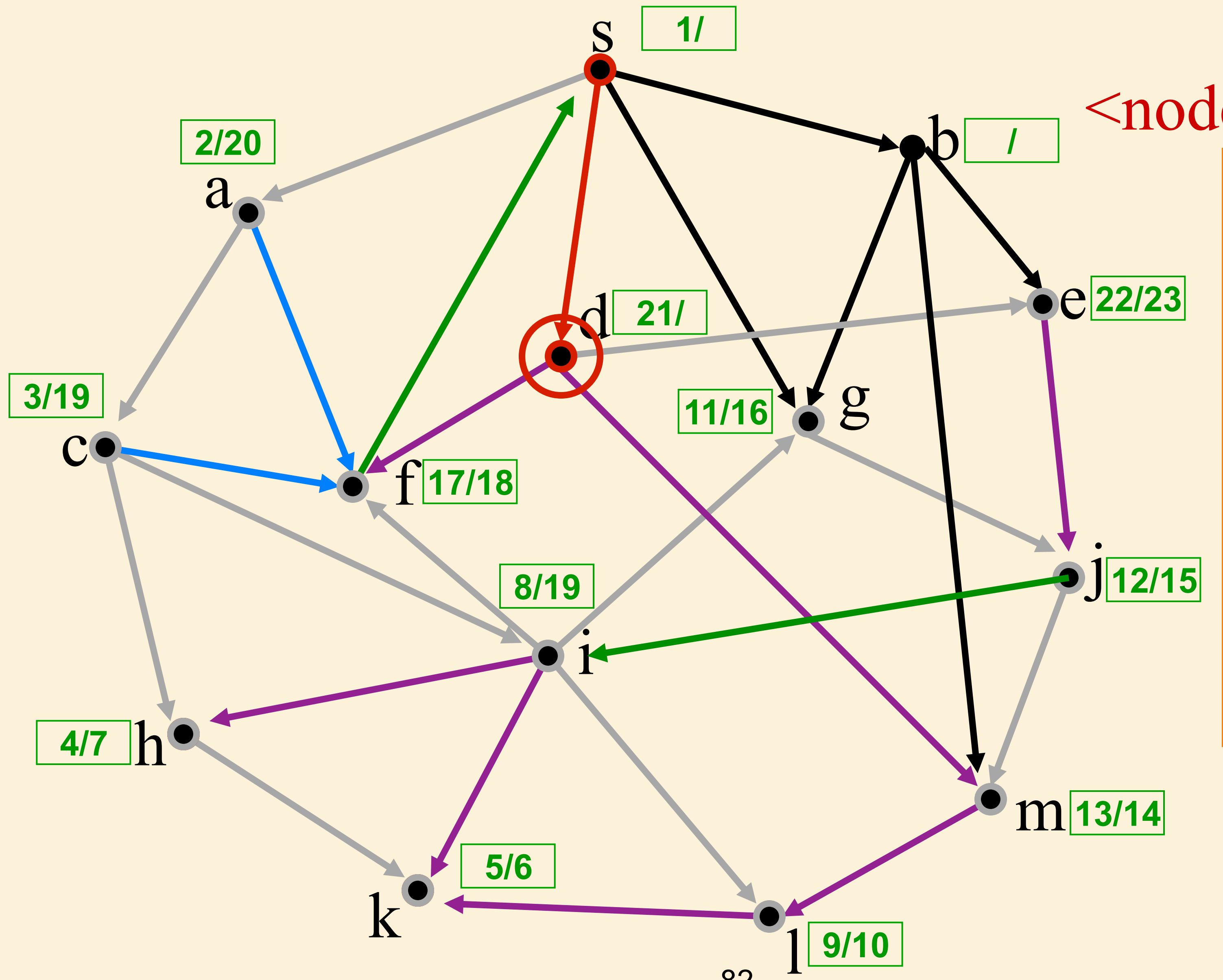
DFS



Found
Not Handled
Stack
<node,# edges>

e,1
d,3
s,2

DFS



Found
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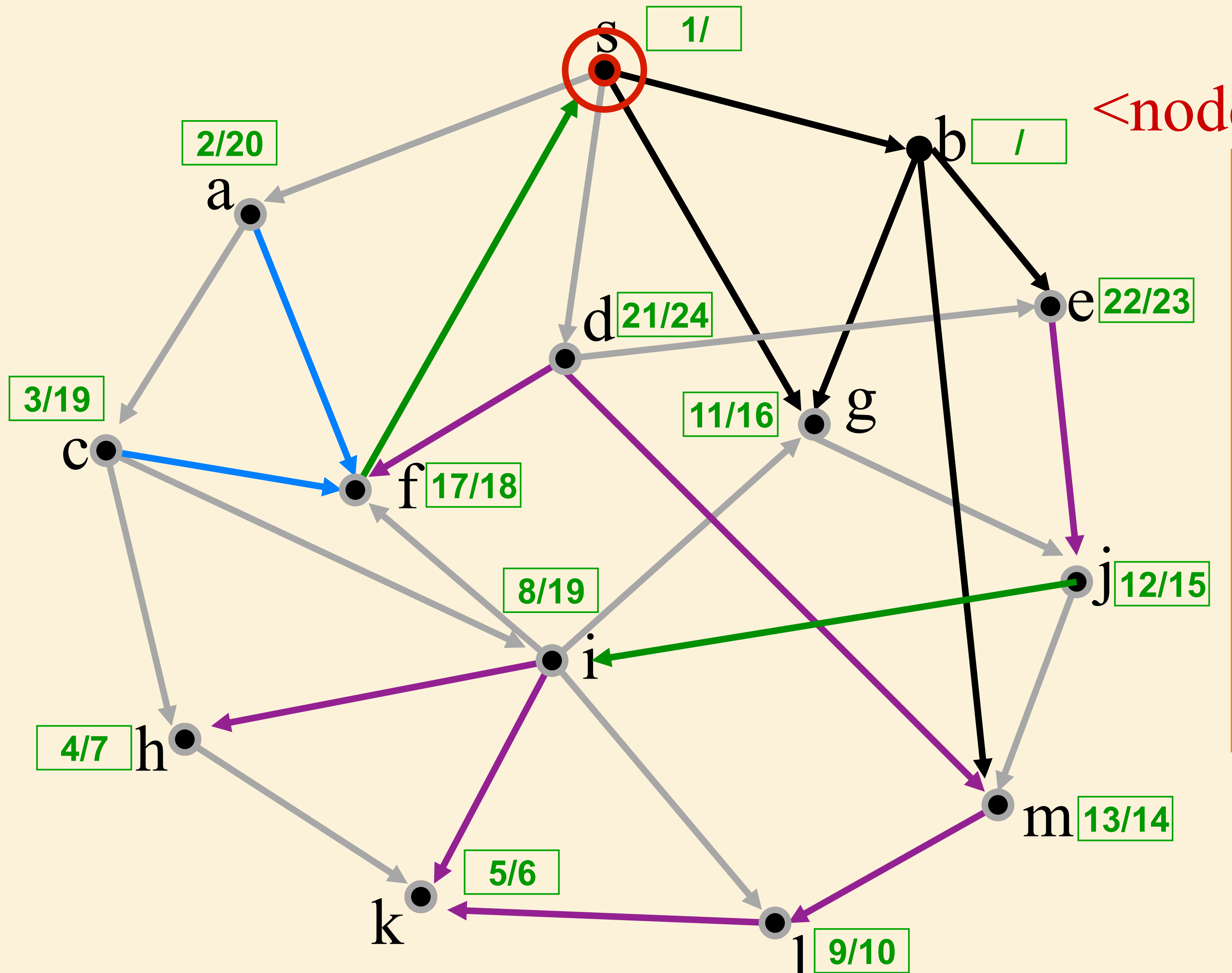
d,3
s,2

DFS



Found
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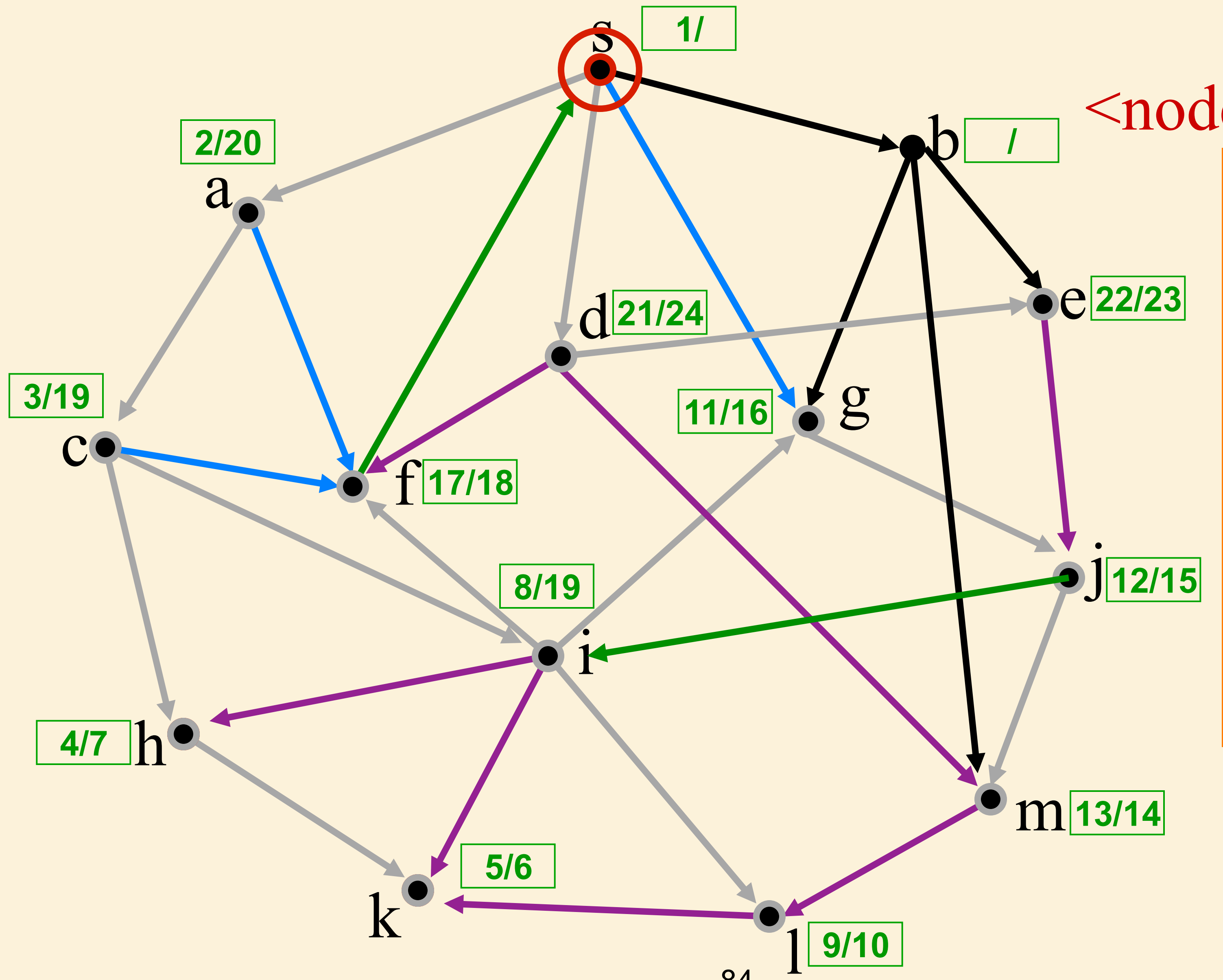
Stack
<node,# edges>

s,2

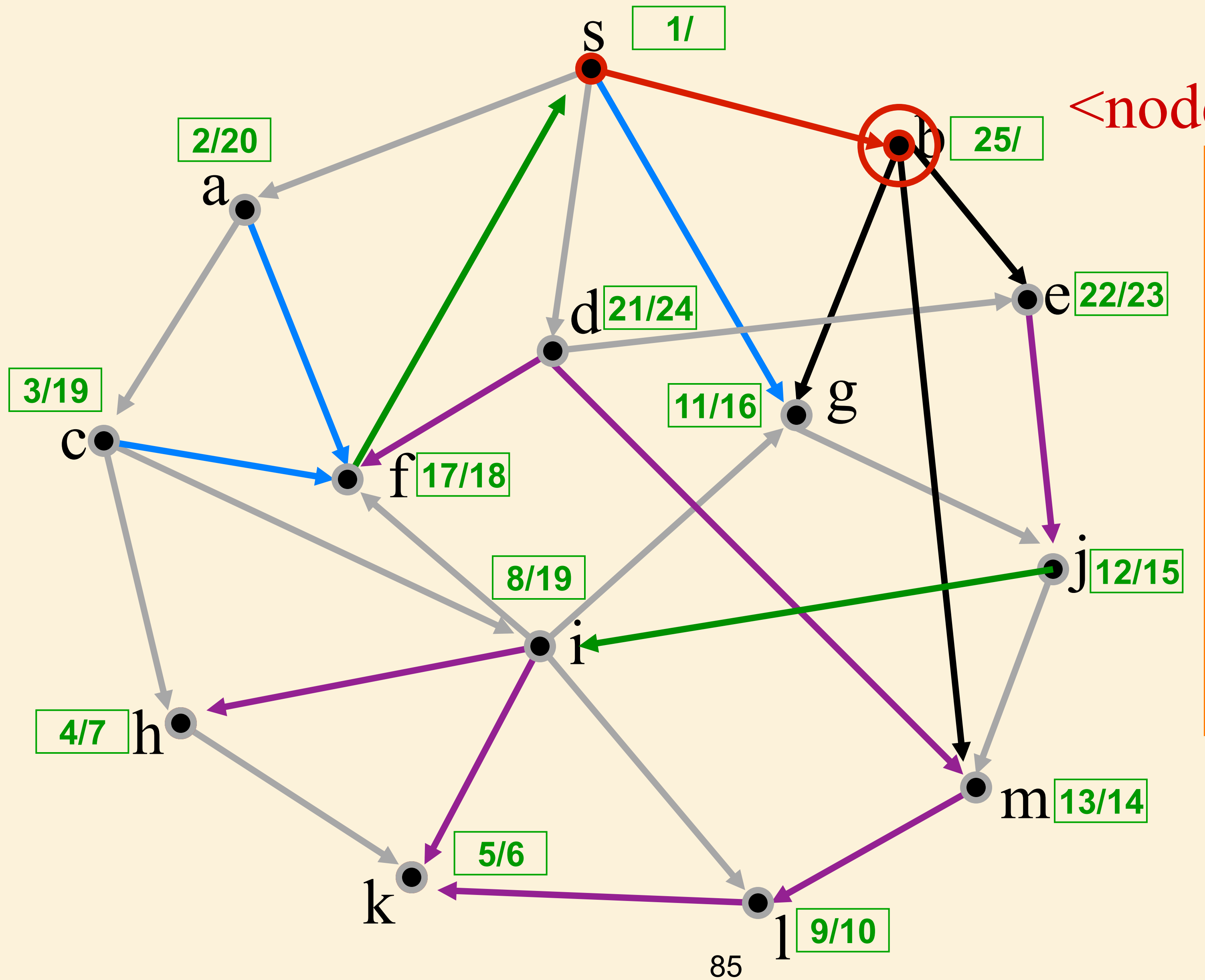
DFS



Found
Not Handled
Stack
<node,# edges>

s,3

DFS



Found

Not Handled

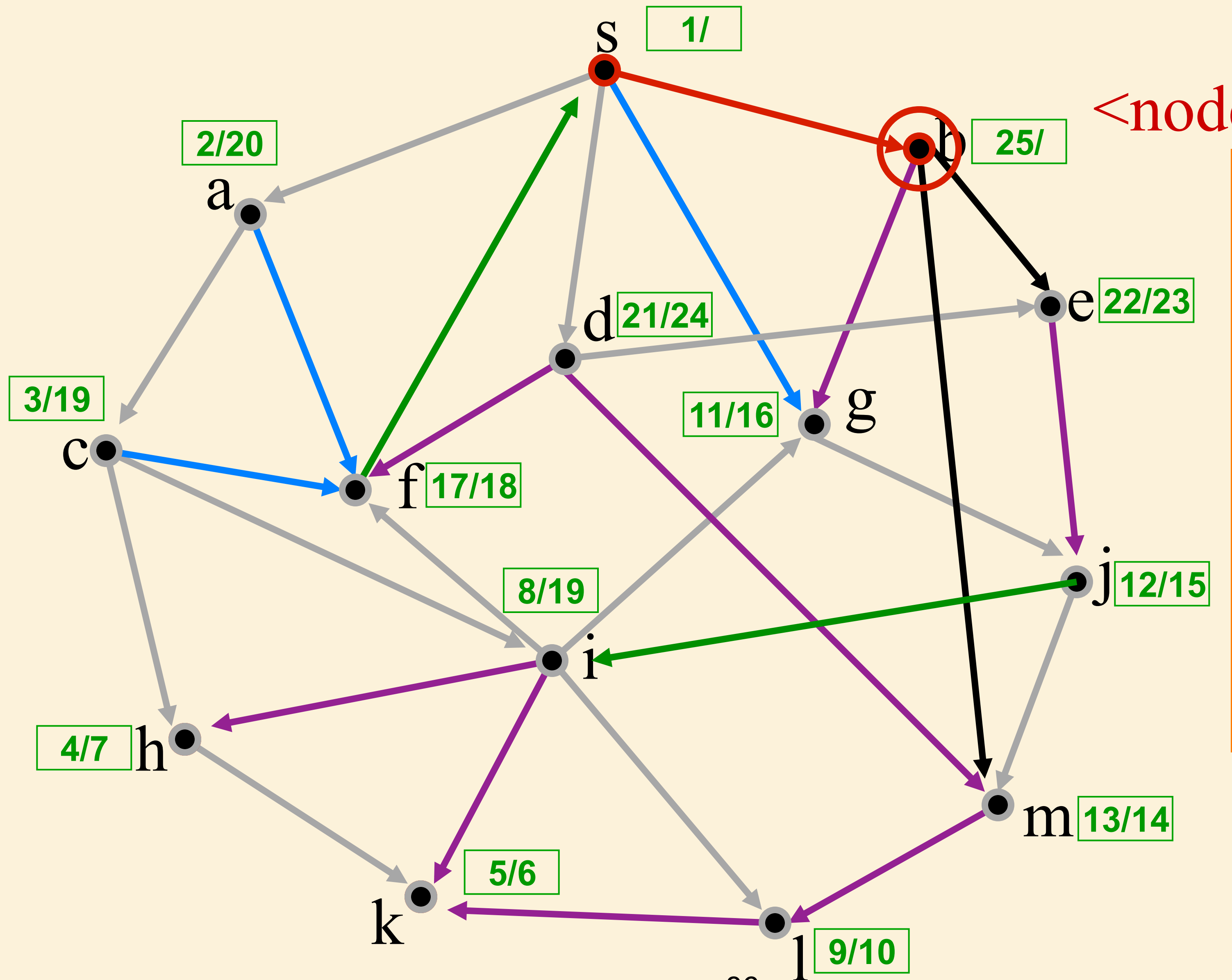
Stack

<node,# edges>

b,0

s,4

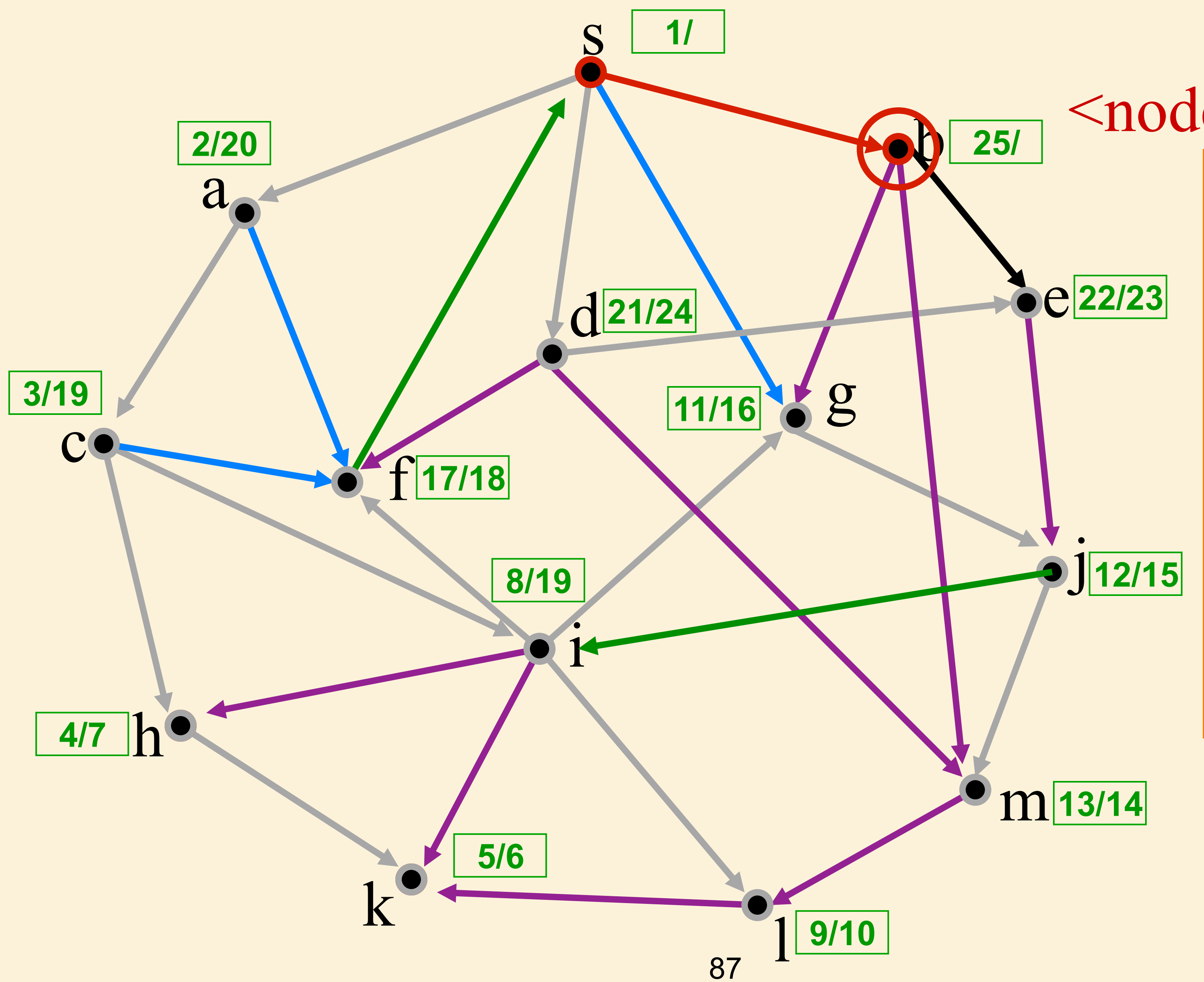
DFS



Found
Not Handled
Stack
<node,# edges>

b,1
s,4

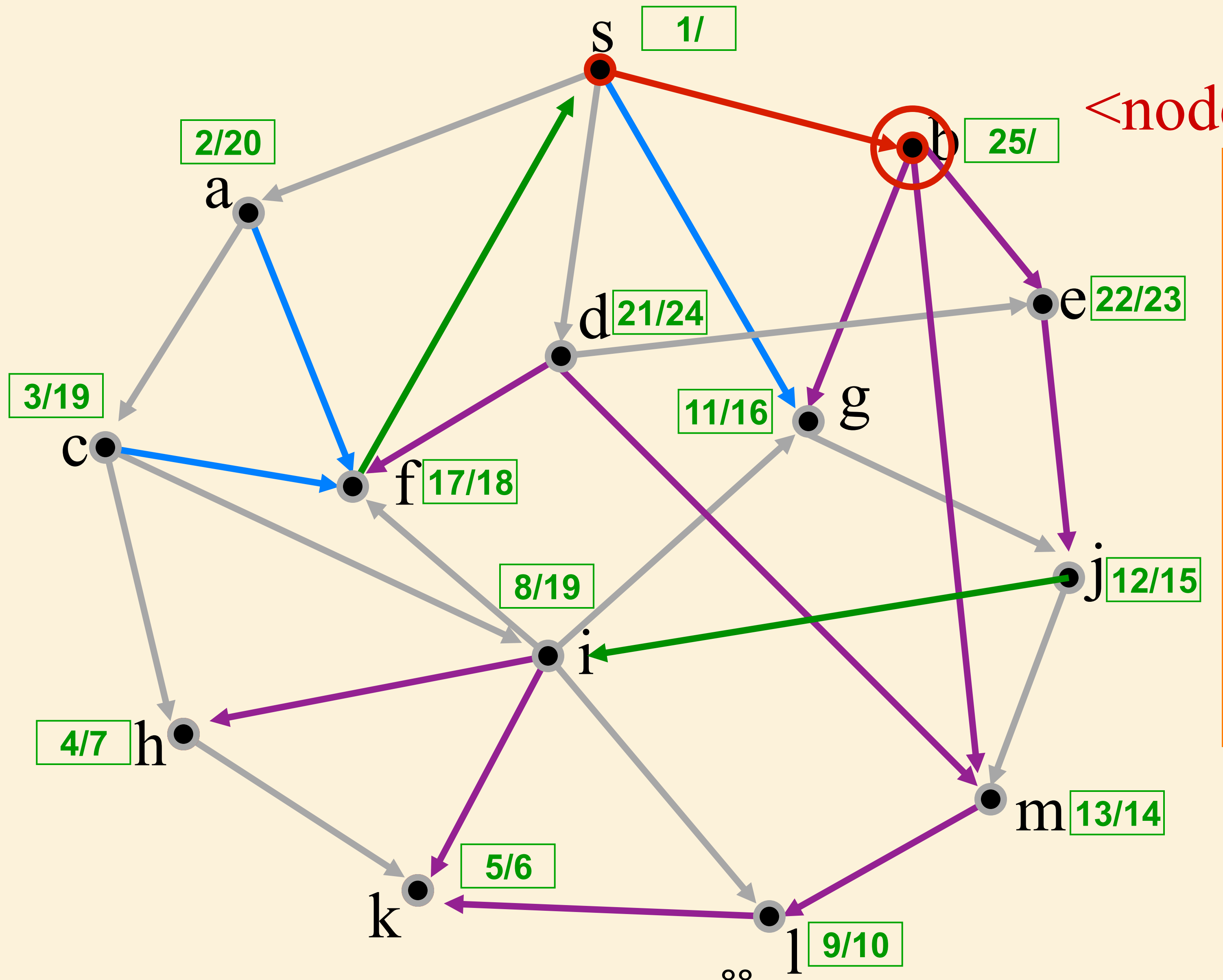
DFS



Found
Not Handled
Stack
<node,# edges>

b,2
s,4

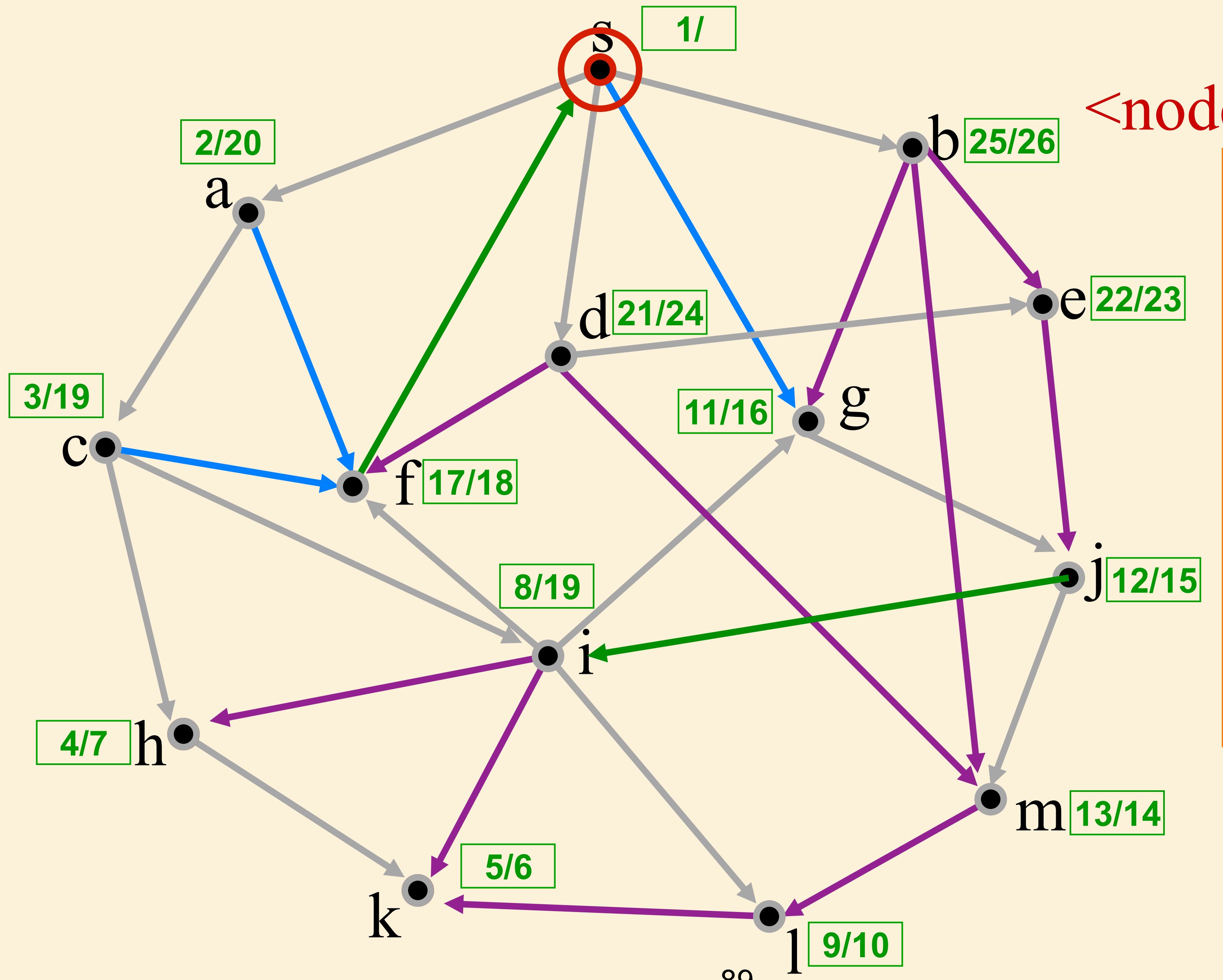
DFS



Found
Not Handled
Stack
<node,# edges>

b,3
s,4

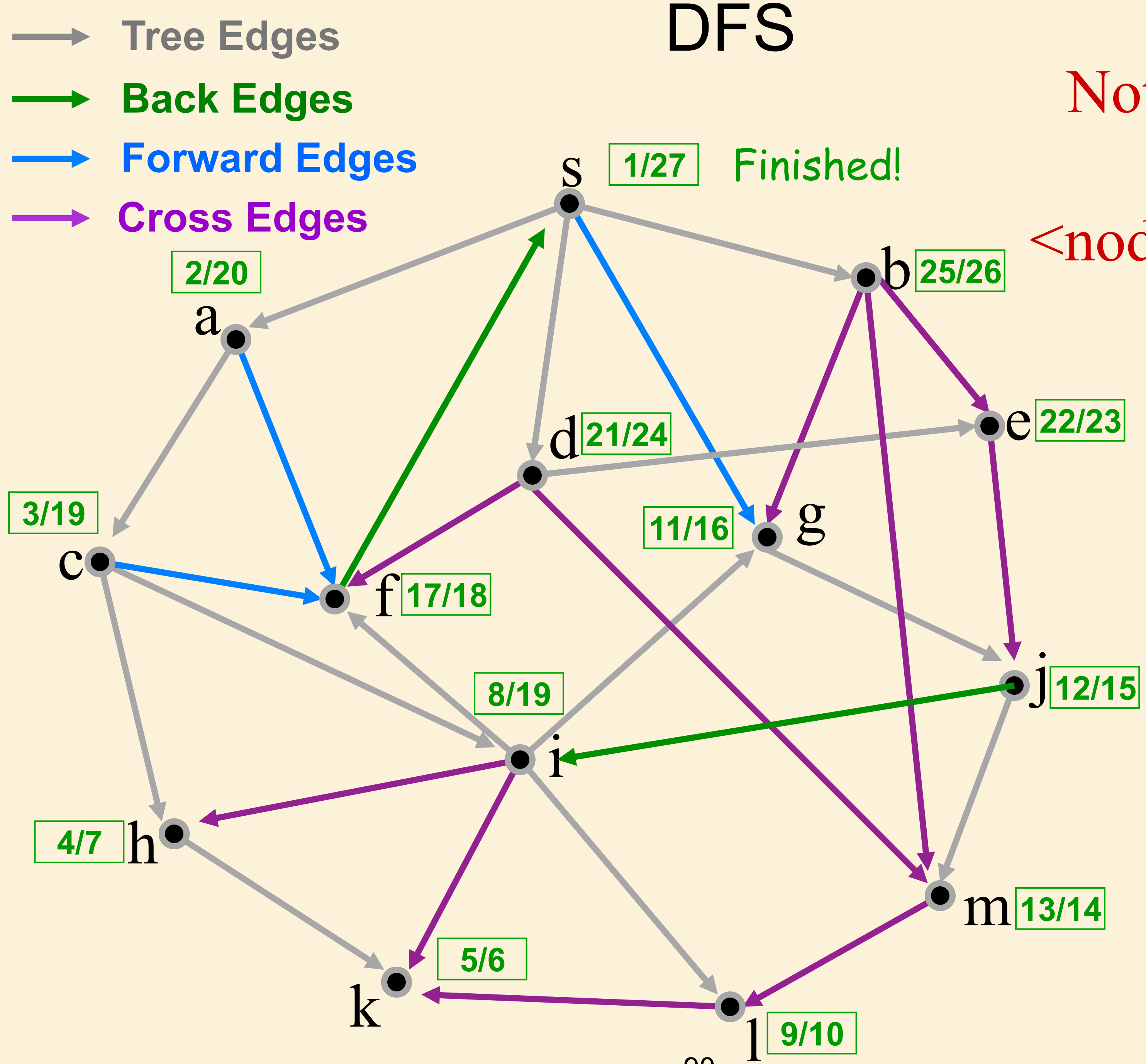
DFS



Found
Not Handled
Stack
<node,# edges>

s,4

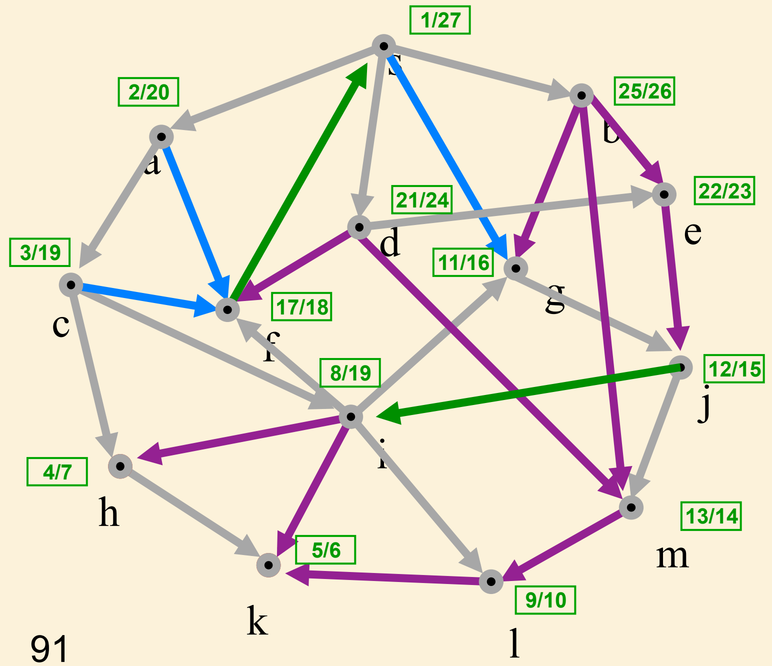
DFS



Found
Not Handled
Stack
<node,# edges>

Classification of Edges in DFS

1. ***Tree edges*** are edges in the depth-first forest G_π. Edge (u, v) is a tree edge if v was first discovered by exploring edge (u, v).
2. ***Back edges*** are those edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree.
3. ***Forward edges*** are non-tree edges (u, v) connecting a vertex u to a descendant v in a depth-first tree.
4. ***Cross edges*** are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other.



Classification of Edges in DFS

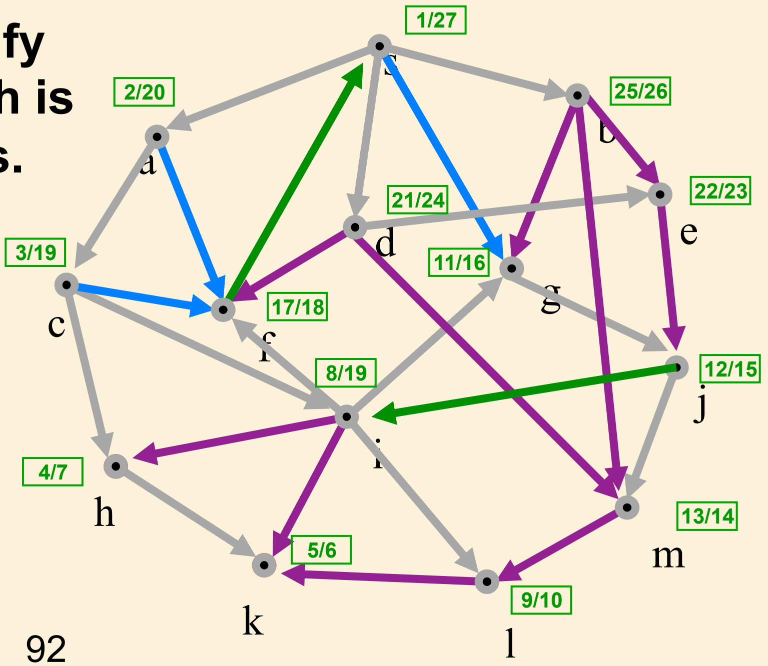
1. ***Tree edges***: Edge (*u*, *v*) is a **tree edge** if *v* was **black** when (*u*, *v*) traversed.

2. ***Back edges:*** (*u*, *v*) is a **back edge** if v was **red** when *(u, v)* traversed.

3. ***Forward edges***: *(u, v)* is a **forward edge** if v was **gray** when *(u, v)* traversed and *d[v] > d[u].*

4. ***Cross edges*** *(u,v)* is a **cross edge** if v was **gray** when *(u, v)* traversed and *d [v] < d[u].*

Classifying edges can help to identify properties of the graph, e.g., a graph is acyclic iff DFS yields no back edges.

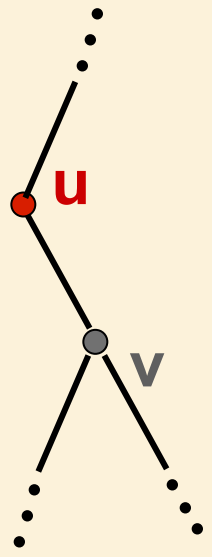


Undirected Graphs

- In a depth-first search of an ***undirected*** graph, every edge is either a **tree edge** or a **back edge**.
- **Why?**

Undirected Graphs

- **Suppose that (u,v)** is a **forward edge** or a **cross edge** in a DFS of an undirected graph.
- **(u,v)** is a **forward edge** or a **cross edge** when **v** is already **handled (grey)** when accessed from **u**.
- This means that all vertices reachable from **v** have been explored.
- Since we are currently handling **u**, **u** must be **red**.
- Clearly **v** is reachable from **u**.
- Since the graph is undirected, **u** must also be reachable from **v**.
- Thus **u** must already have been handled: **u** must be **grey**.
- **Contradiction!**



Depth-First Search Algorithm

DFS(G)

```
1  for each vertex $u \in V[G]$
2       do $color[u] \leftarrow$ BLACK
3          $\pi[u] \leftarrow$ NIL
4  $time \leftarrow 0$
5  for each vertex $u \in V[G]$
6       do if $color[u] =$ BLACK
7             then DFS-VISIT($u$)
```

DFS-Visit (u)

Precondition: vertex u is undiscovered

Postcondition: all vertices reachable from u have been processed

```
1  $color[u] \leftarrow$ RED              ▷BLACK vertex $u$ has just been discovered.
2  $time \leftarrow time + 1$
3  $d[u] \leftarrow time$
4  for each $v \in Adj[u]$            ▷ Explore edge $(u, v)$.
5       do if $color[v] =$ BLACK
6             then $\pi[v] \leftarrow u$
7                  DFS-VISIT($v$)
8  $color[u] \leftarrow$ GRAY            ▷   GRAY $u$; it is finished.
9  $f[u] \leftarrow time \leftarrow time + 1$
```

Depth-First Search Algorithm

DFS(G)

```
for each vertex $u \in V[G]$
    do $color[u] \leftarrow$ BLACK
        $\pi[u] \leftarrow$ NIL
$time \leftarrow 0$
for each vertex $u \in V[G]$
    do if $color[u] =$ BLACK
        then DFS-VISIT($u$)
```

total work = $\theta(V)$

Thus running time = $\theta(V+E)$

DFS-Visit (u)

Precondition: vertex u is undiscovered

Postcondition: all vertices reachable from u have been processed

```
$color[u] \leftarrow$ RED          ▷BLACK vertex $u$ has just been discovered.
$time \leftarrow time + 1$
$d[u] \leftarrow time$
for each $v \in Adj[u]$          ▷ Explore edge $(u, v)$.
    do if $color[v] =$ BLACK
        then $\pi[v] \leftarrow u$
            DFS-VISIT($v$)
$color[u] \leftarrow$ GRAY          ▷ GRAY $u$; it is finished.
$f[u] \leftarrow time \leftarrow time + 1$
```

total work = $\sum_{v \in V} |Adj[v]| = \theta(E)$

Topological Sorting
(e.g., putting tasks in linear order)

An application of Depth-First Search

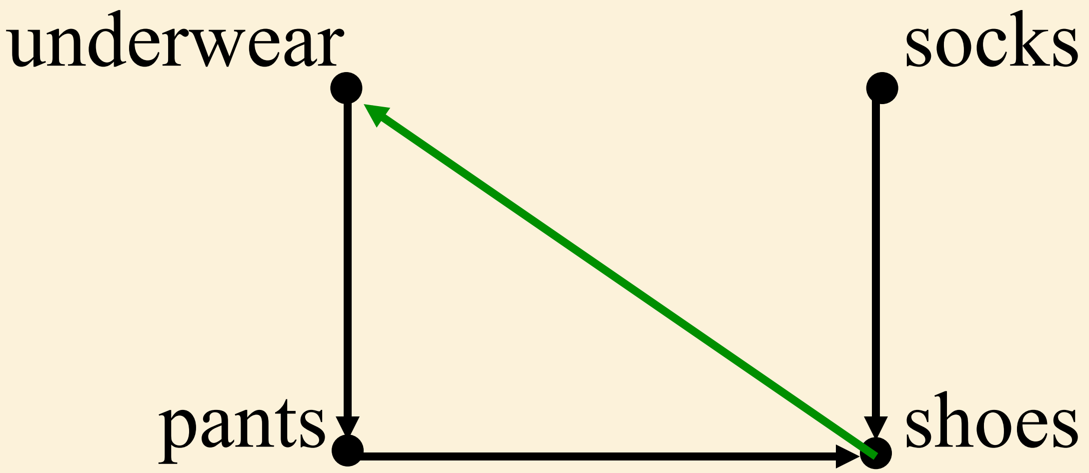
Linear Order

$$\begin{tikzcd}
\text{underwear} \arrow[d] & & \text{socks} \arrow[d] \\
\text{pants} \arrow[rr] & & \text{shoes}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{underwear} \arrow[d, bend left] \\
\text{pants} \arrow[dd, bend left] \\
\text{socks} \arrow[d, bend left] \\
\text{shoes}
\end{tikzcd}$$

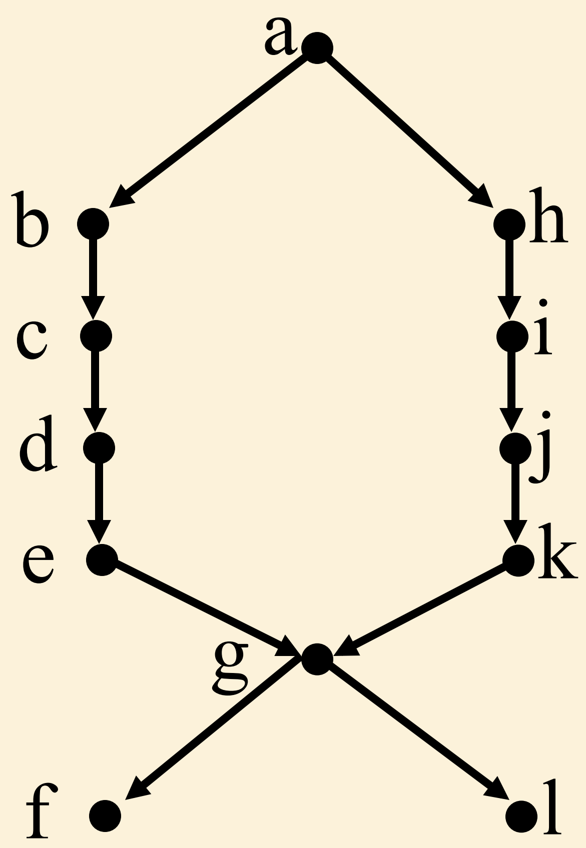
$$\begin{tikzcd}
\text{socks} \arrow[ddd, bend left] \\
\text{underwear} \arrow[d, bend left] \\
\text{pants} \arrow[d, bend left] \\
\text{shoes}
\end{tikzcd}$$

Linear Order



Too many video games?

Linear Order



Precondition:

A Directed Acyclic Graph (DAG)

Post Condition:

Find one valid linear order

Algorithm:

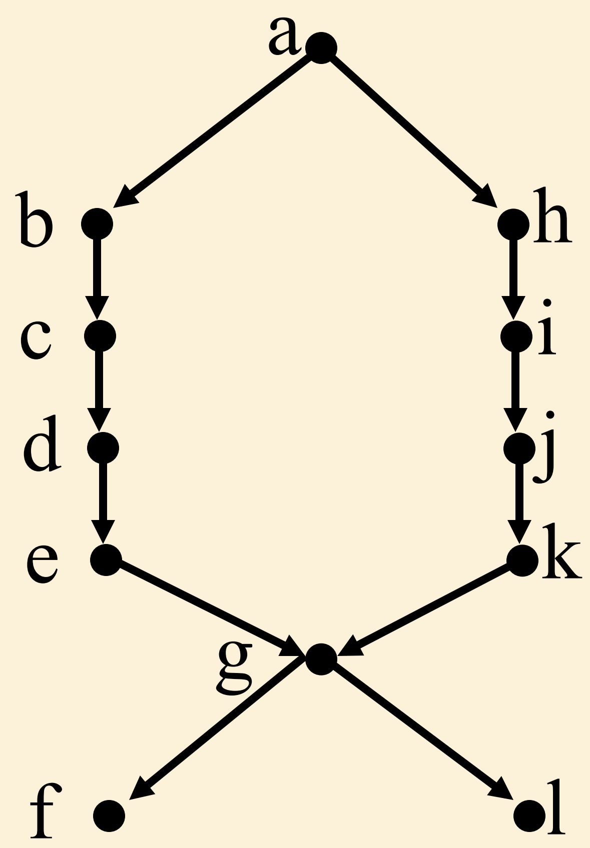
- Find a terminal node (sink). ~~$\Theta(V)$~~
- Put it last in sequence.
- Delete from graph & repeat ~~$\Theta(V^2)$~~

We can do better!

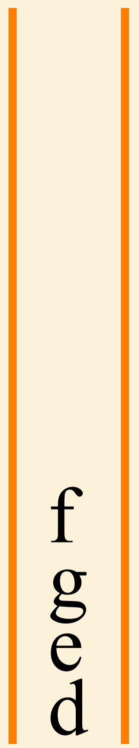
….. l

Linear Order

Alg: DFS



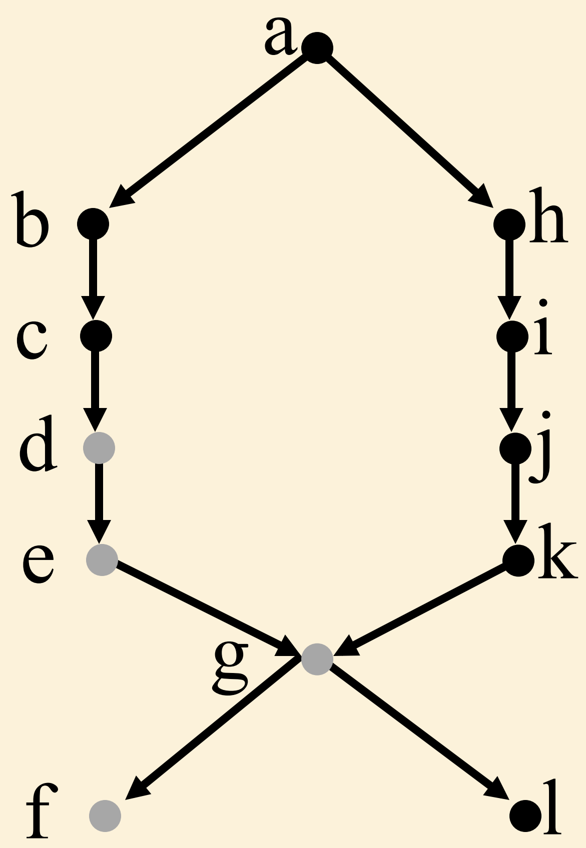
Found
Not Handled
Stack



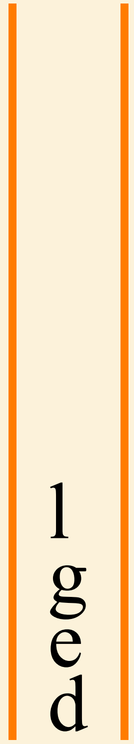
….. f

Linear Order

Alg: DFS



Found
Not Handled
Stack



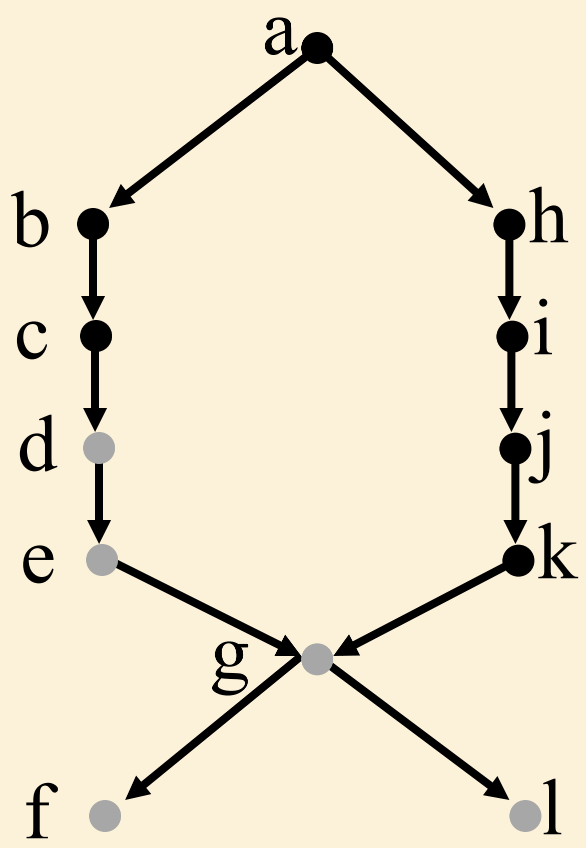
When node is popped off stack, insert at front of linearly-ordered “to do” list.

Linear Order:

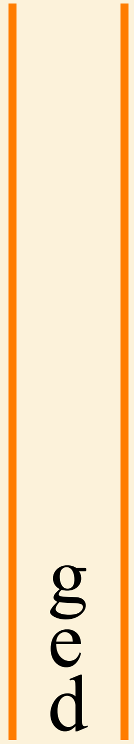
….. f

Linear Order

Alg: DFS



Found
Not Handled
Stack

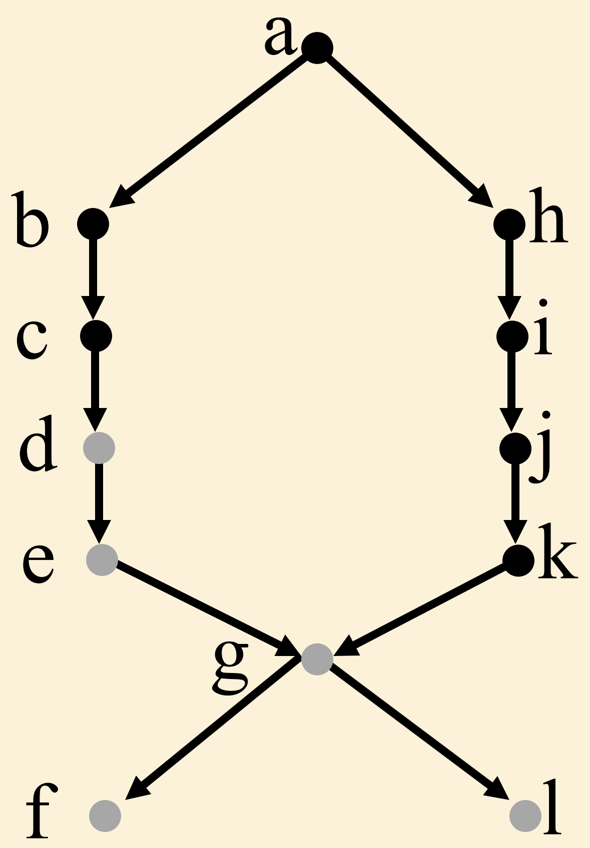


Linear Order:

l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

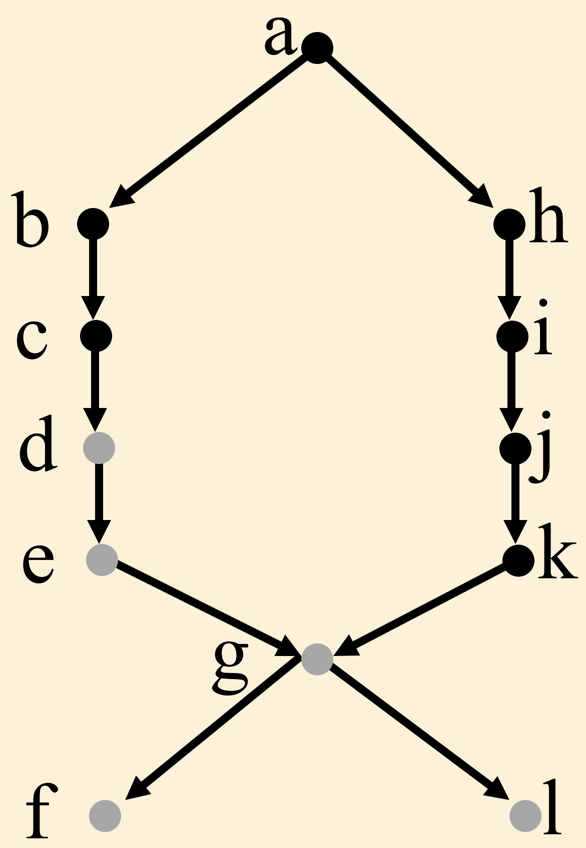
e
d

Linear Order:

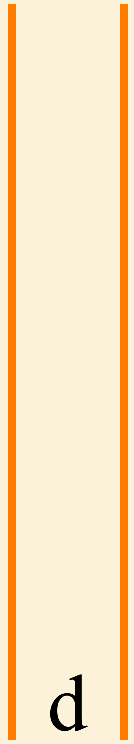
g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

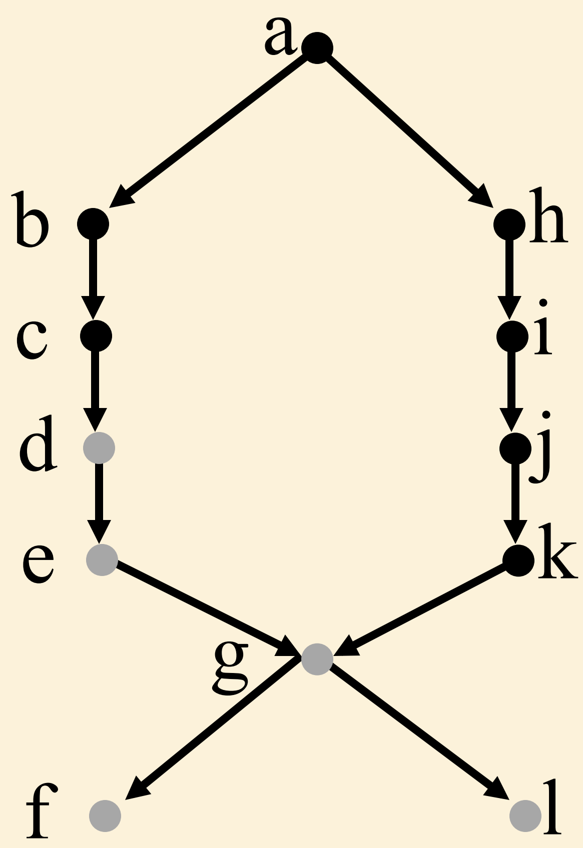


Linear Order:

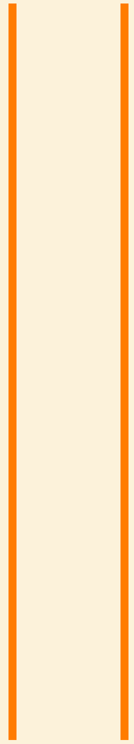
e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

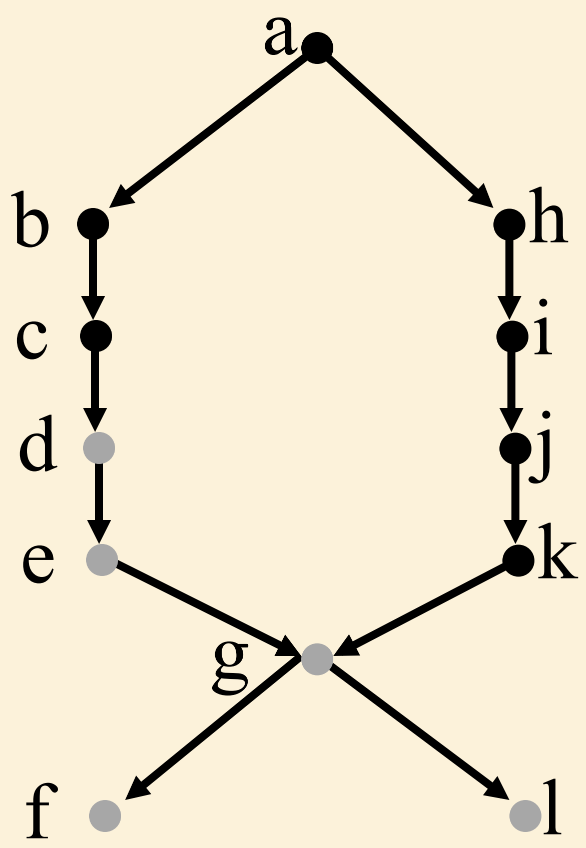


Linear Order:

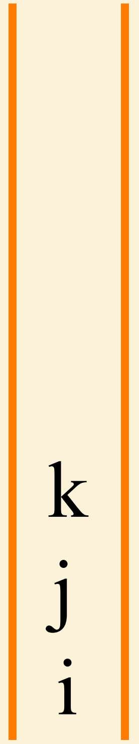
d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

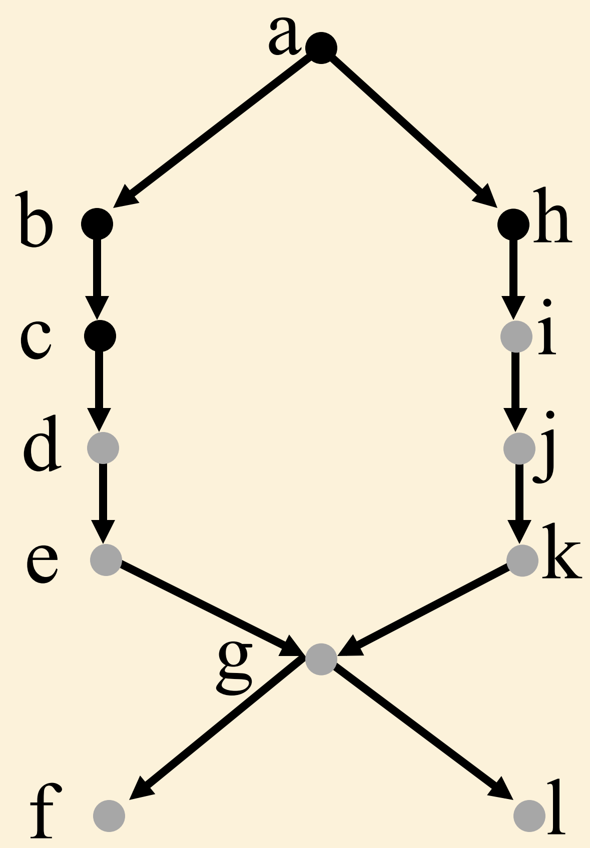


Linear Order:

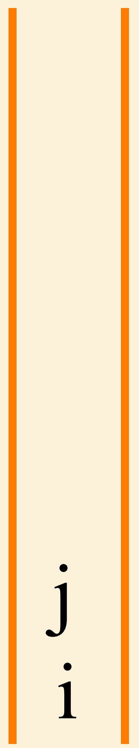
d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

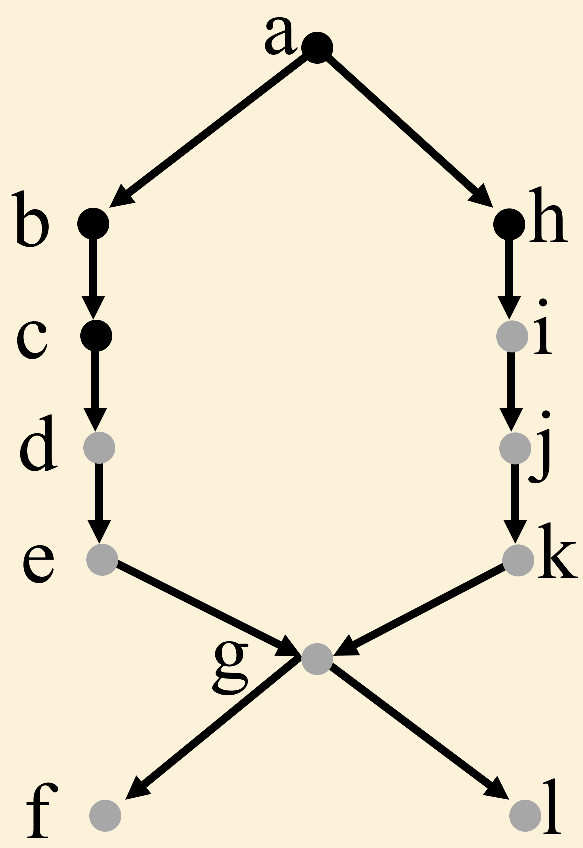


Linear Order:

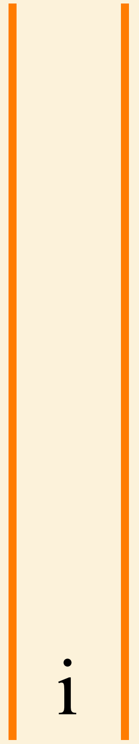
k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

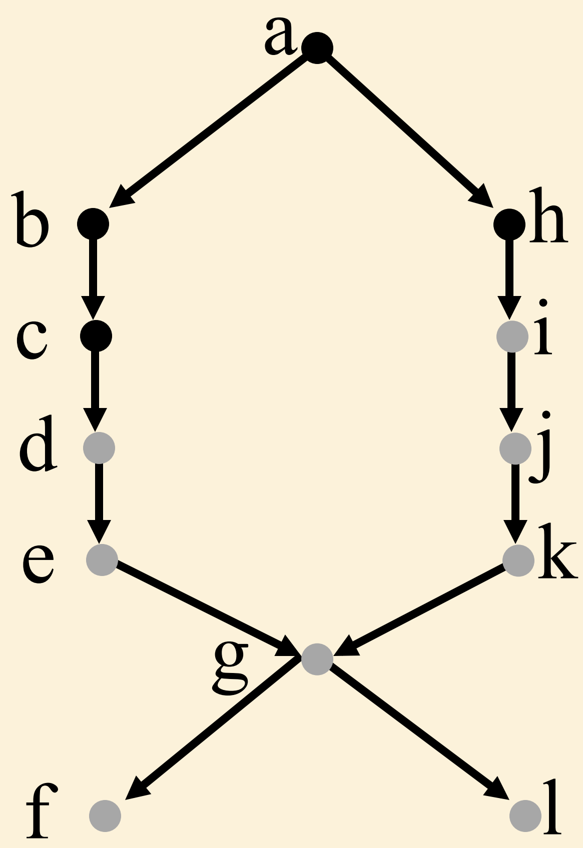


Linear Order:

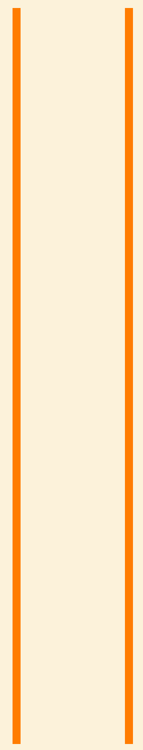
j,k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

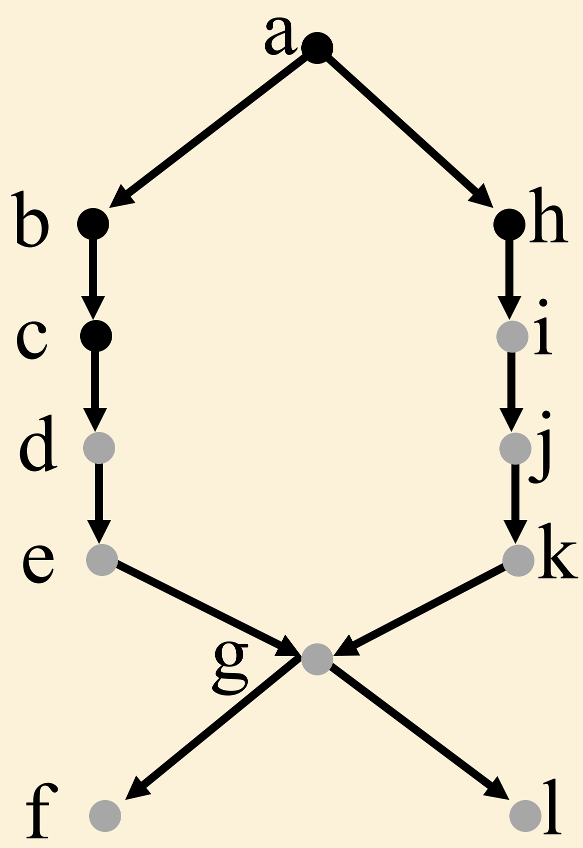


Linear Order:

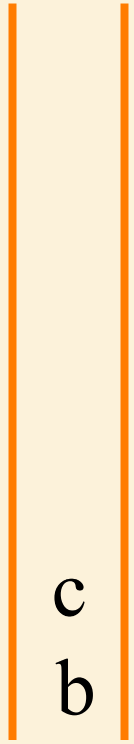
i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

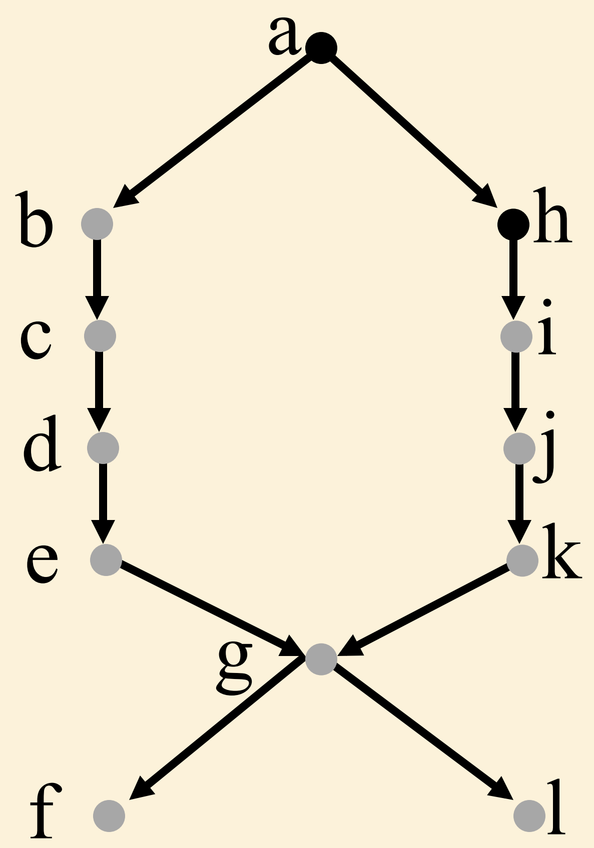


Linear Order:

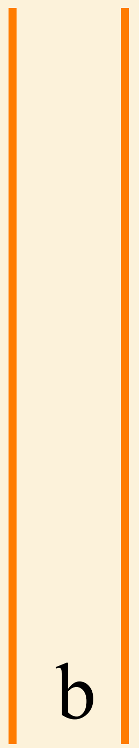
i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

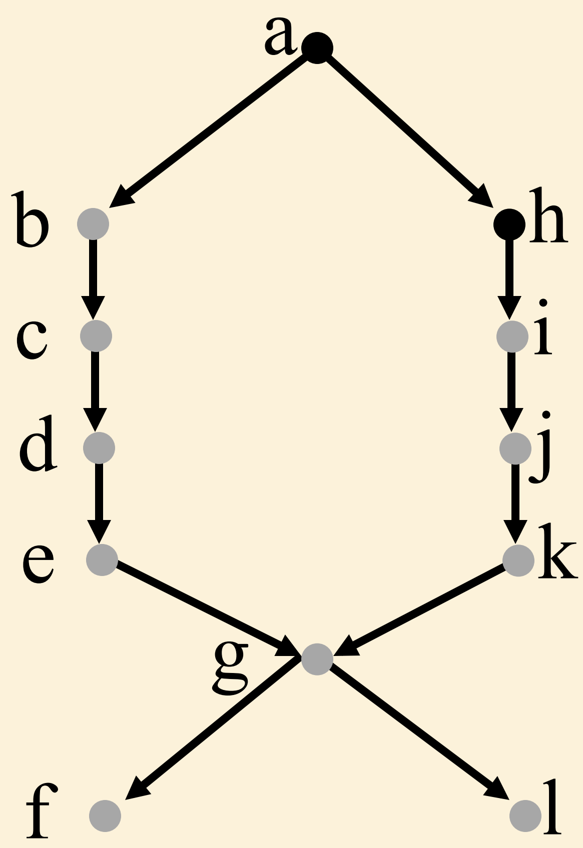


Linear Order:

c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



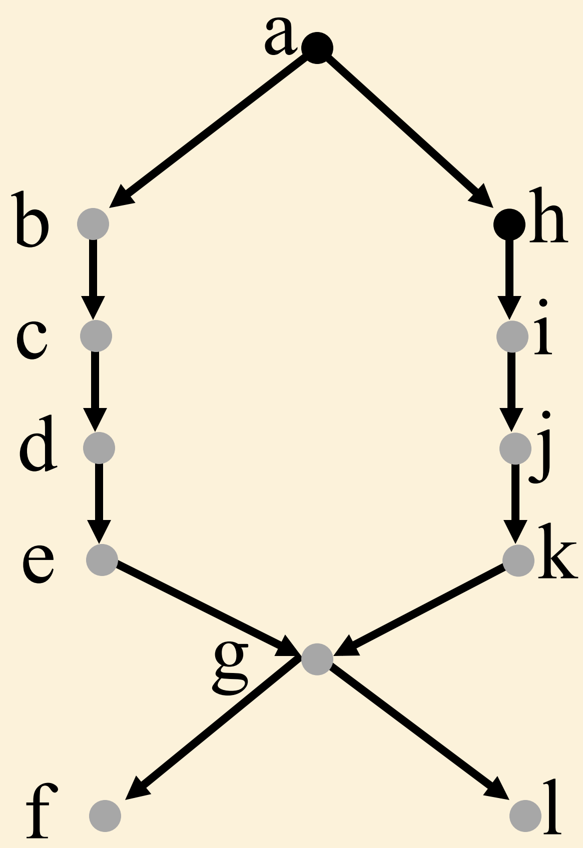
Found
Not Handled
Stack

Linear Order:

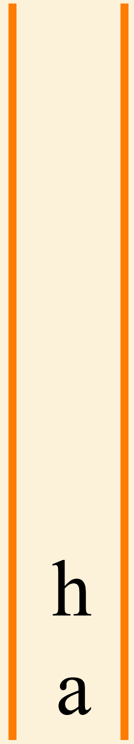
b,c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

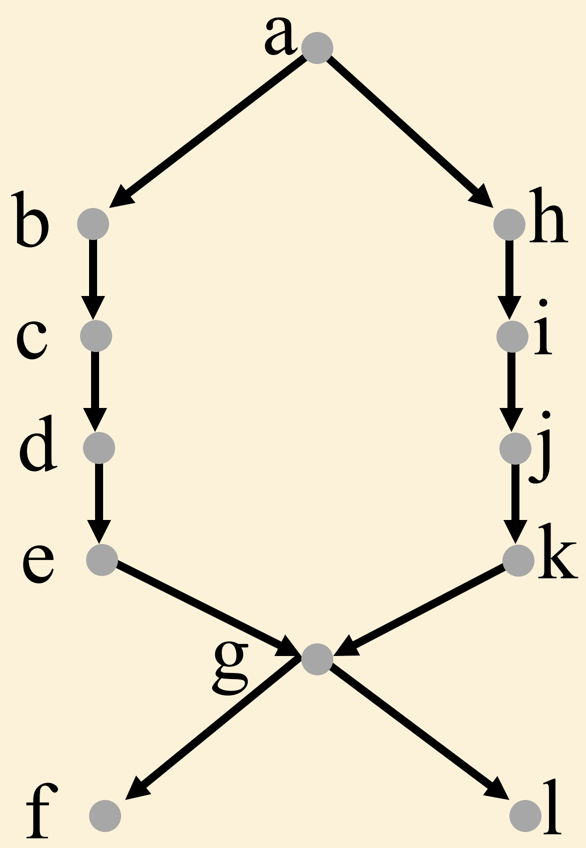


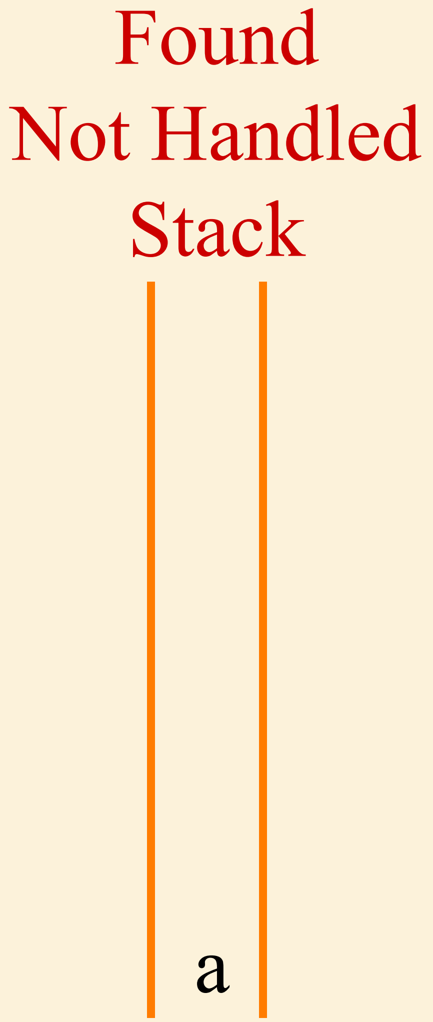
Linear Order:

b,c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



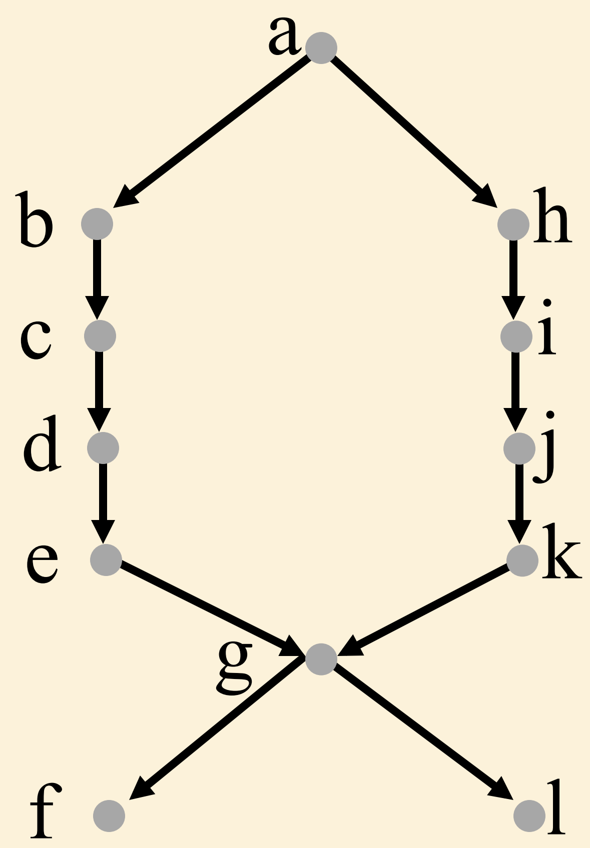


Linear Order:

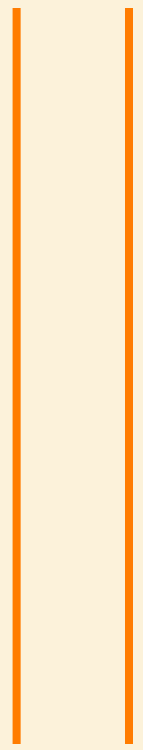
h,b,c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack



Linear Order:

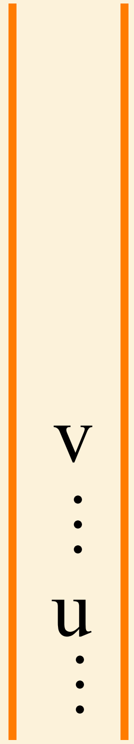
a,h,b,c,i,j,k,d,e,g,l,f **Done!**

Linear Order

Proof: Consider each edge

- Case 1: u goes on stack first before v.
  - Because of edge,
    v goes on before u comes off
  - v comes off before u comes off
  - v goes after u in order. ☺

Found
Not Handled
Stack



$$\begin{tikzcd} u \arrow[r] & v \end{tikzcd}$$

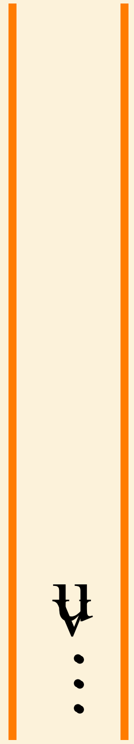
u…v…

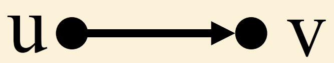
Linear Order

Proof: Consider each edge

- Case 1: u goes on stack first before v.
- Case 2: v goes on stack first before u.
  v comes off before u goes on.
  - v goes after u in order. ☺

Found
Not Handled
Stack





u…v…

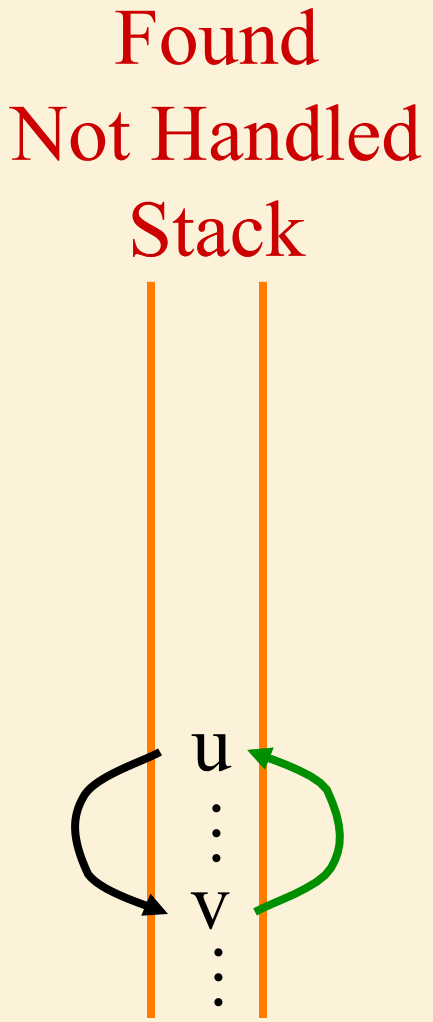
Linear Order

Proof: Consider each edge

- Case 1: u goes on stack first before v.
- Case 2: v goes on stack first before u.
  v comes off before u goes on.

Case 3: v goes on stack first before u.
  u goes on before v comes off.

- Panic: u goes after v in order. ☹
- Cycle means linear order
  is impossible ☺



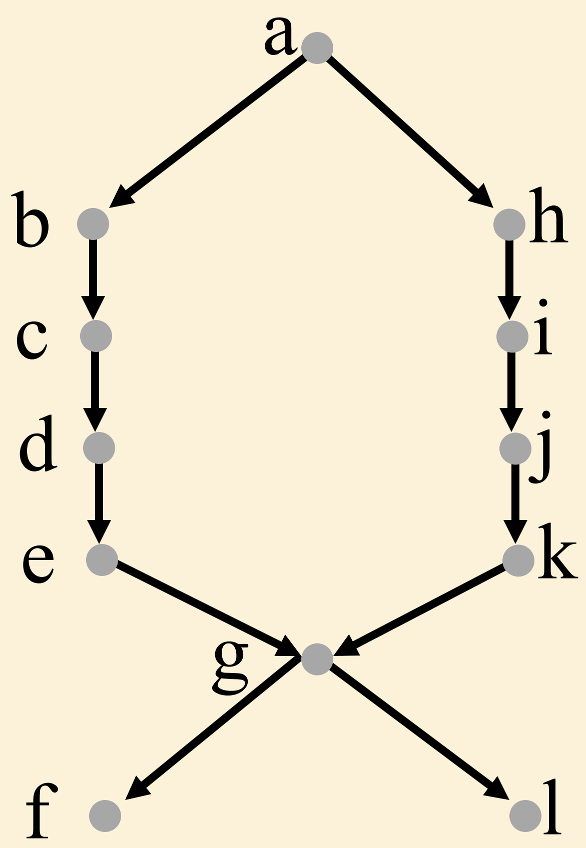
The nodes in the stack form a path starting at s.

$$\begin{tikzcd} u \arrow[r] & v \end{tikzcd}$$

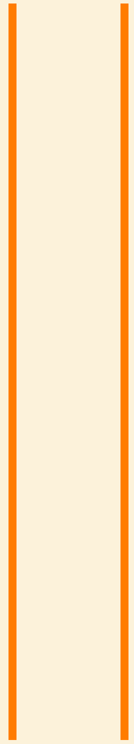
v…u…

Linear Order

Alg: DFS



Found
Not Handled
Stack



Analysis: $\Theta(V+E)$

Linear Order:

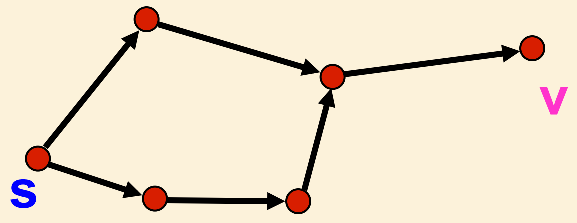
a,h,b,c,i,j,k,d,e,g,l,f **Done!**

Shortest Paths Revisited

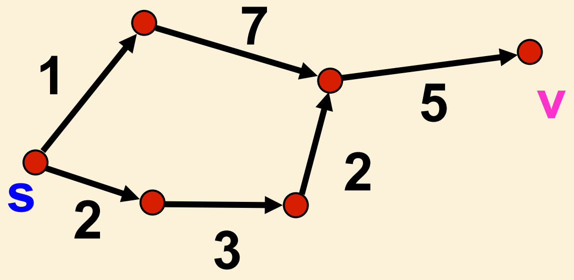
Back to Shortest Path

- BFS finds the **shortest paths** from a source node **s** to every vertex **v** in the graph.
- Here, the **length** of a path is simply the number of edges on the path.
- But what if edges have different ‘costs’?

$\delta(s,v) = 3$



$\delta(s,v) = 12$



Single-Source (Weighted) Shortest Paths

The Problem

- What is the shortest driving route from Toronto to Ottawa? (e.g. MAPQuest, Google Maps)
- Input:

Directed Graph $G = (V, E)$

Edge weights $w : E \to {}^{\circ}$

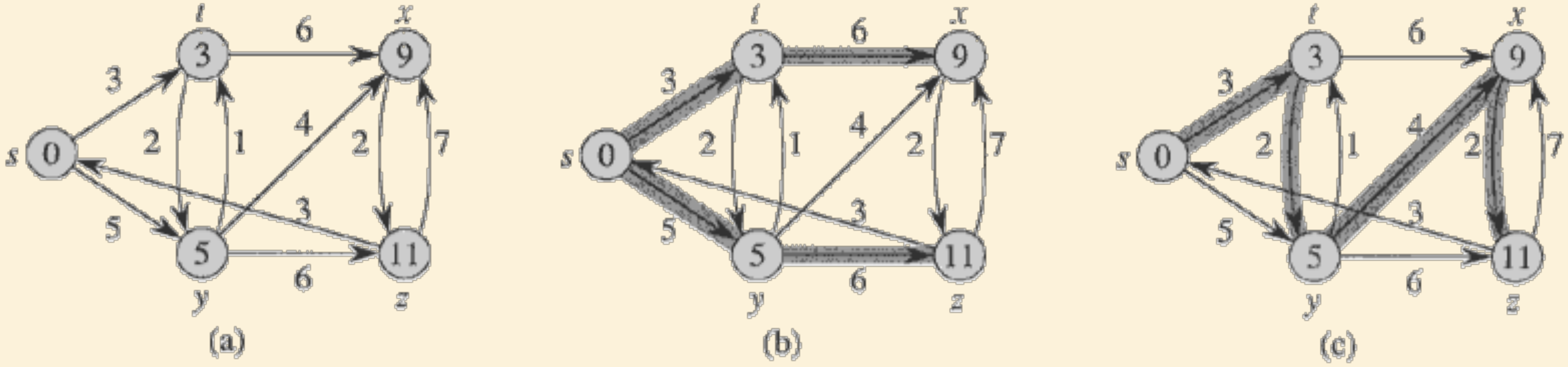
Weight of path $p = < v_0, v_1, \ldots, v_k > \; = \sum_{i=1}^{k} w(v_{i-1}, v_i)$

Shortest-path weight from u to v:

$$\delta(u,v) = \begin{cases} \min\{w(p) : \; u \overset{p}{\to \mathrm{L}} \to v\} & \text{if } \exists \text{ a path } u \to \mathrm{L} \to v, \\ \infty & \text{otherwise.} \end{cases}$$

Shortest path from u to v is any path p such that $w(p) = \delta(u,v)$.

Example



Single-source shortest path search induces a search tree rooted at *s*.

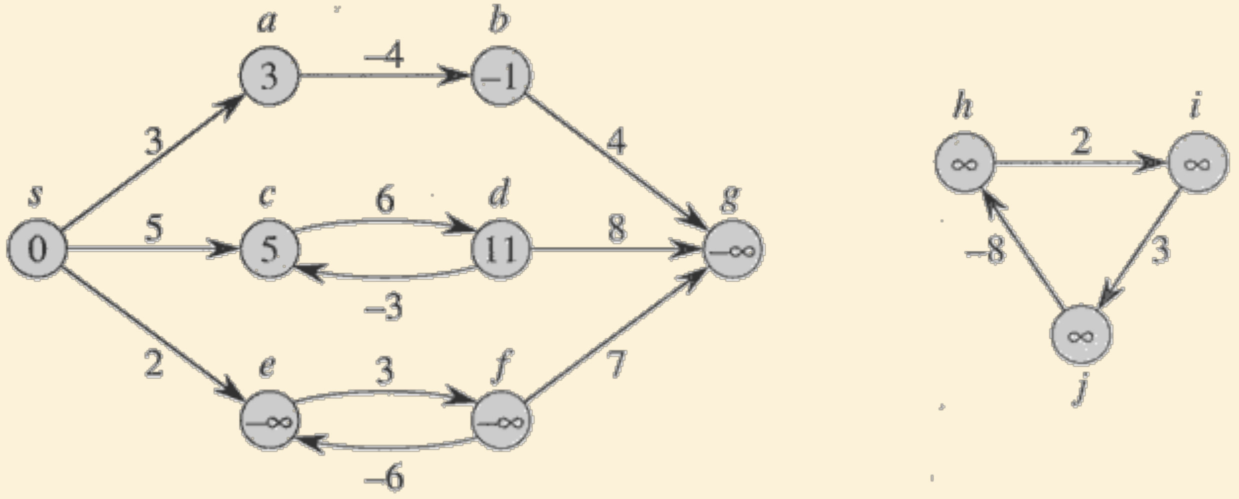
This tree, and hence the paths themselves, are not necessarily unique.

Shortest path variants

- **Single-source shortest-paths problem:** – the shortest path from *s* to each vertex *v*. (e.g. BFS)
- **Single-destination shortest-paths problem:** Find a shortest path to a given ***destination*** vertex *t* from each vertex *v*.
- **Single-pair shortest-path problem:** Find a shortest path from *u* to *v* for given vertices *u* and *v*.
- **All-pairs shortest-paths problem:** Find a shortest path from *u* to *v* for every pair of vertices *u* and *v*.

Negative-weight edges

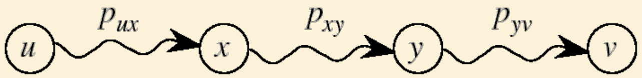
- OK, as long as no negative-weight cycles are reachable from the source.
  - If we have a negative-weight cycle, we can just keep going around it, and get w(s, v) = −∞ for all v on the cycle.
  - But OK if the negative-weight cycle is not reachable from the source.
  - Some algorithms work only if there are no negative-weight edges in the graph.



Optimal substructure

- Lemma: Any subpath of a shortest path is a shortest path
- Proof: Cut and paste.

Suppose this path p is a shortest path from u to v.

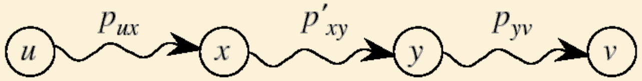


Then $\delta(u,v) = w(p) = w(p_{ux}) + w(p_{xy}) + w(p_{yv})$.

Now suppose there exists a shorter path $x \rightarrow \overset{p'_{xy}}{\mathrm{L}} \rightarrow y$.

Then $w(p'_{xy}) < w(p_{xy})$.

Construct p′:



Then $w(p') = w(p_{ux}) + w(p'_{xy}) + w(p_{yv}) < w(p_{ux}) + w(p_{xy}) + w(p_{yv}) = w(p)$.

So p wasn't a shortest path after all!

Cycles

- Shortest paths can’t contain cycles:
  - Already ruled out negative-weight cycles.
  - Positive-weight: we can get a shorter path by omitting the cycle.
  - Zero-weight: no reason to use them → assume that our solutions won’t use them.

Output of a single-source shortest-path algorithm

- For each vertex v in V:
  - d[v] = δ(s, v).
    - Initially, d[v]=∞.
    - Reduce as algorithm progresses.
      But always maintain d[v] ≥ δ(s, v).
    - Call d[v] a shortest-path estimate.
  - π[v] = predecessor of v on a shortest path from s.
    - If no predecessor, π[v] = NIL.
    - π induces a tree — **shortest-path tree**.

Initialization

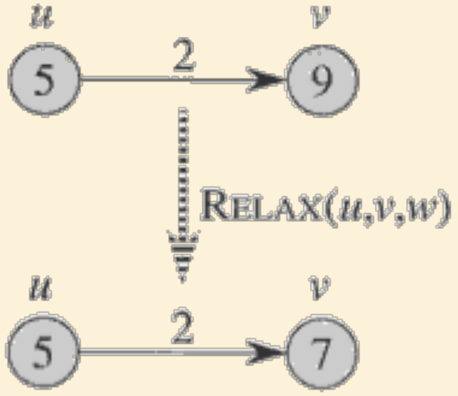
- All shortest-paths algorithms start with the same initialization:

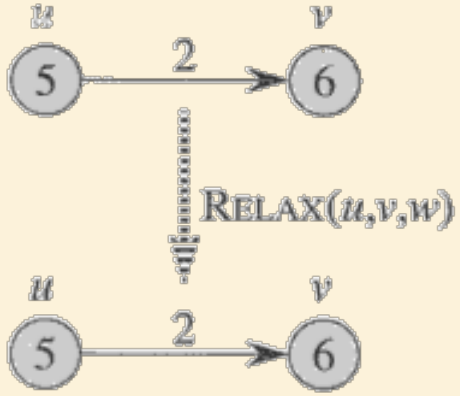
```
INIT-SINGLE-SOURCE(V, s)
for each v in V
    do d[v] ← ∞
       π[v] ← NIL
d[s] ← 0
```

Relaxing an edge

- Can we improve shortest-path estimate for v by going through u and taking (u,v)?

```
RELAX(u, v,w)
    if $d[v] > d[u] + w(u, v)$ then
        $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$
```





General single-source shortest-path strategy

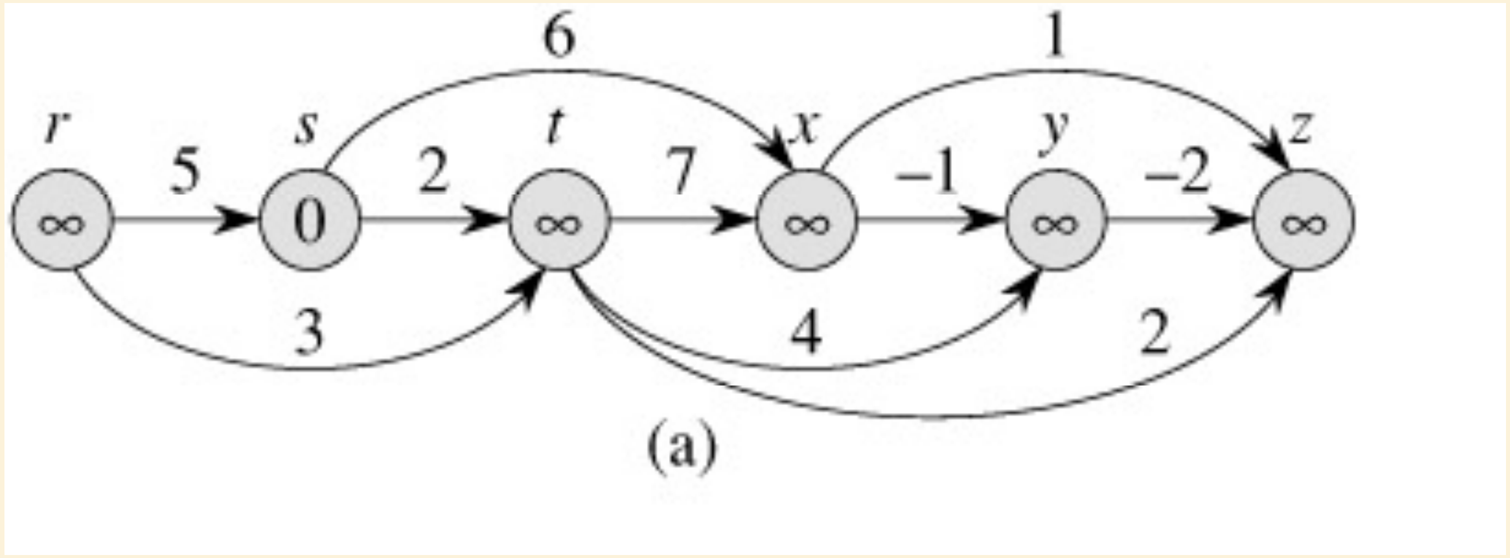
1. Start by calling INIT-SINGLE-SOURCE
2. Relax Edges

Algorithms differ in the order in which edges are taken and

how many times each edge is relaxed.

Example: Single-source shortest paths in a directed acyclic graph (DAG)

- Since graph is a DAG, we are guaranteed no negative-weight cycles.

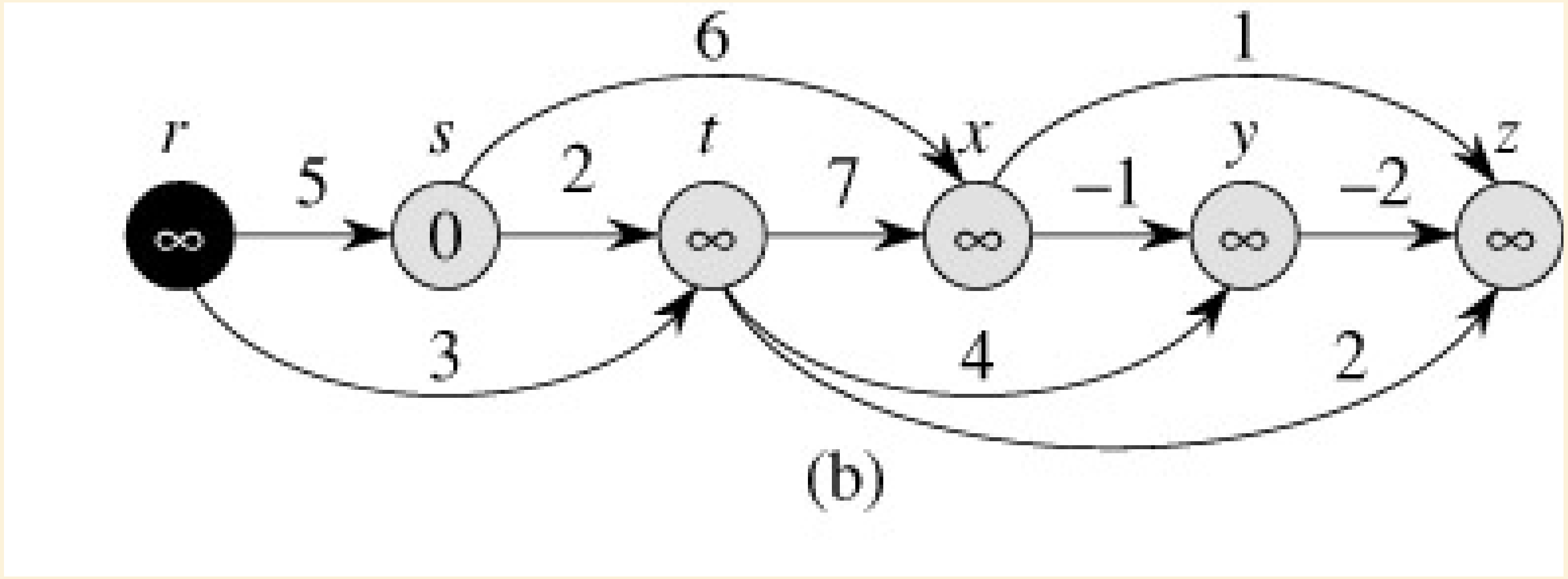


Algorithm

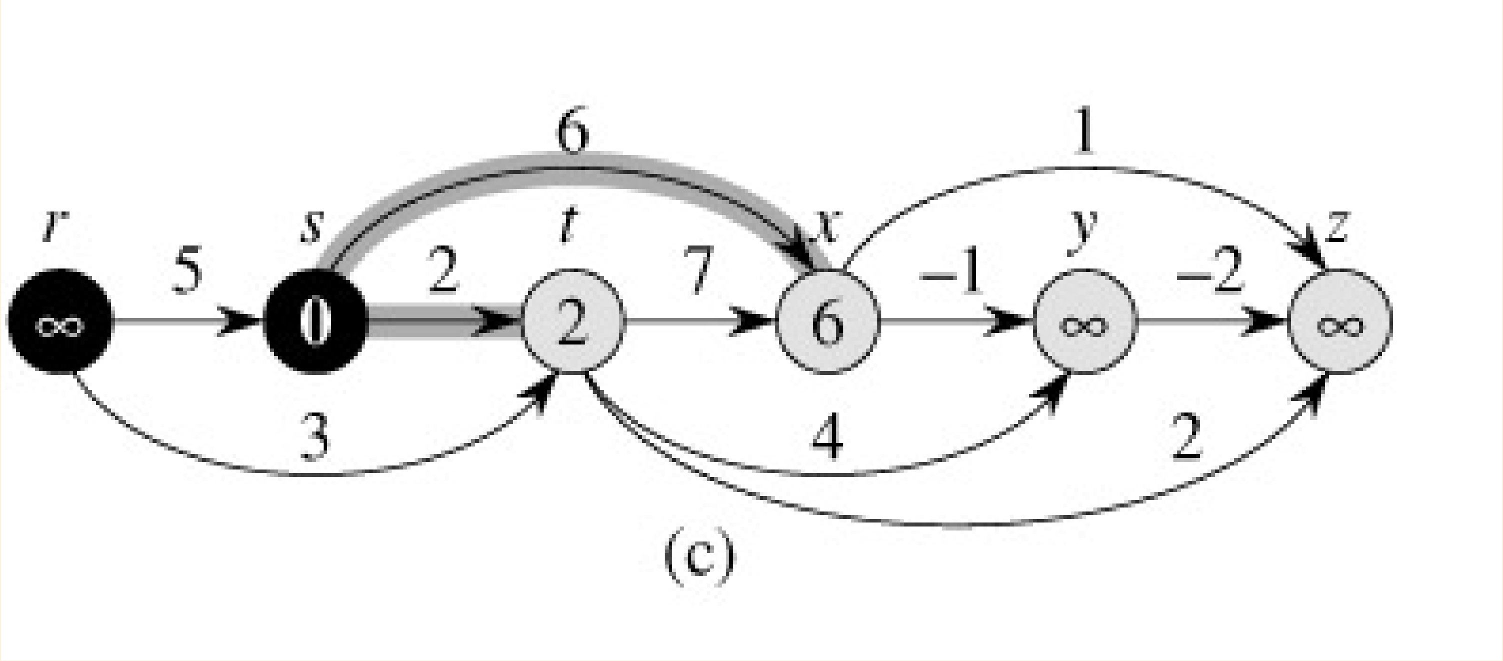
```
DAG-SHORTEST-PATHS$(G, w, s)$
1  topologically sort the vertices of $G$
2  INITIALIZE-SINGLE-SOURCE$(G, s)$
3  for each vertex $u$, taken in topologically sorted order
4       do for each vertex $v \in Adj[u]$
5              do RELAX$(u, v, w)$
```

Time: $\Theta(V + E)$

Example

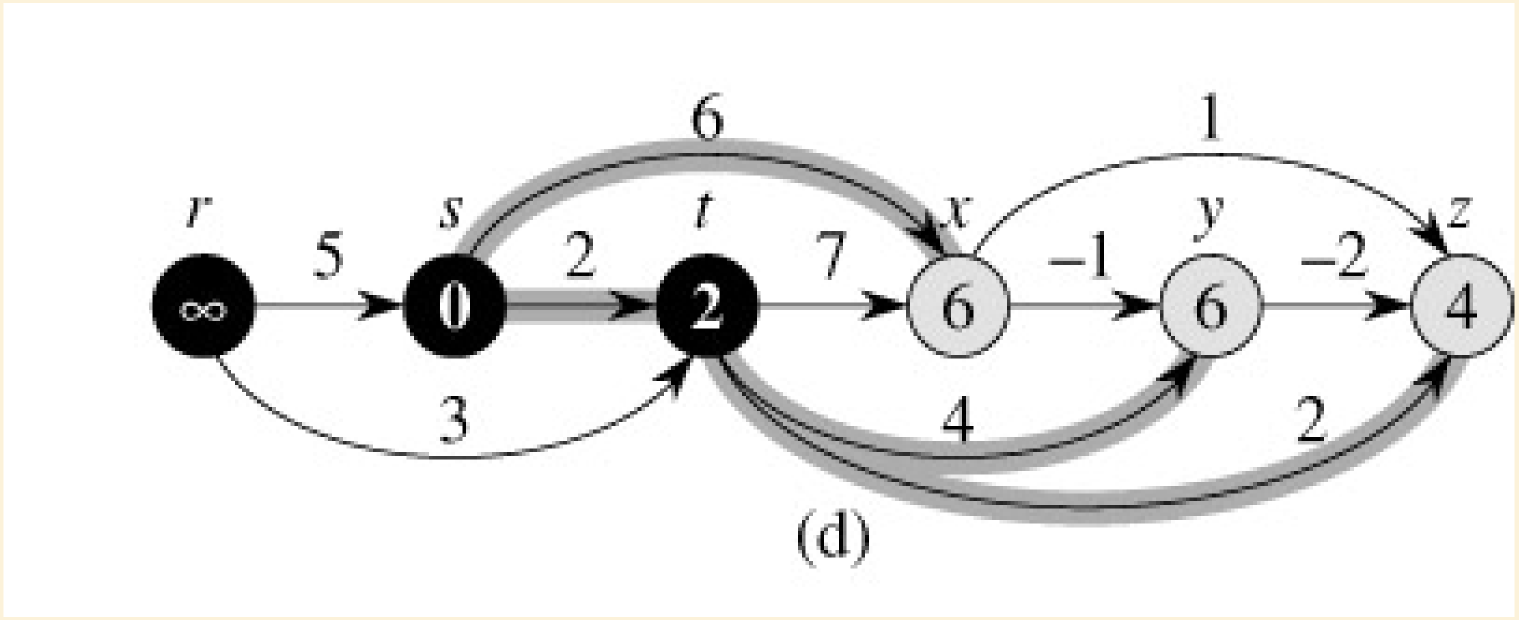


Example



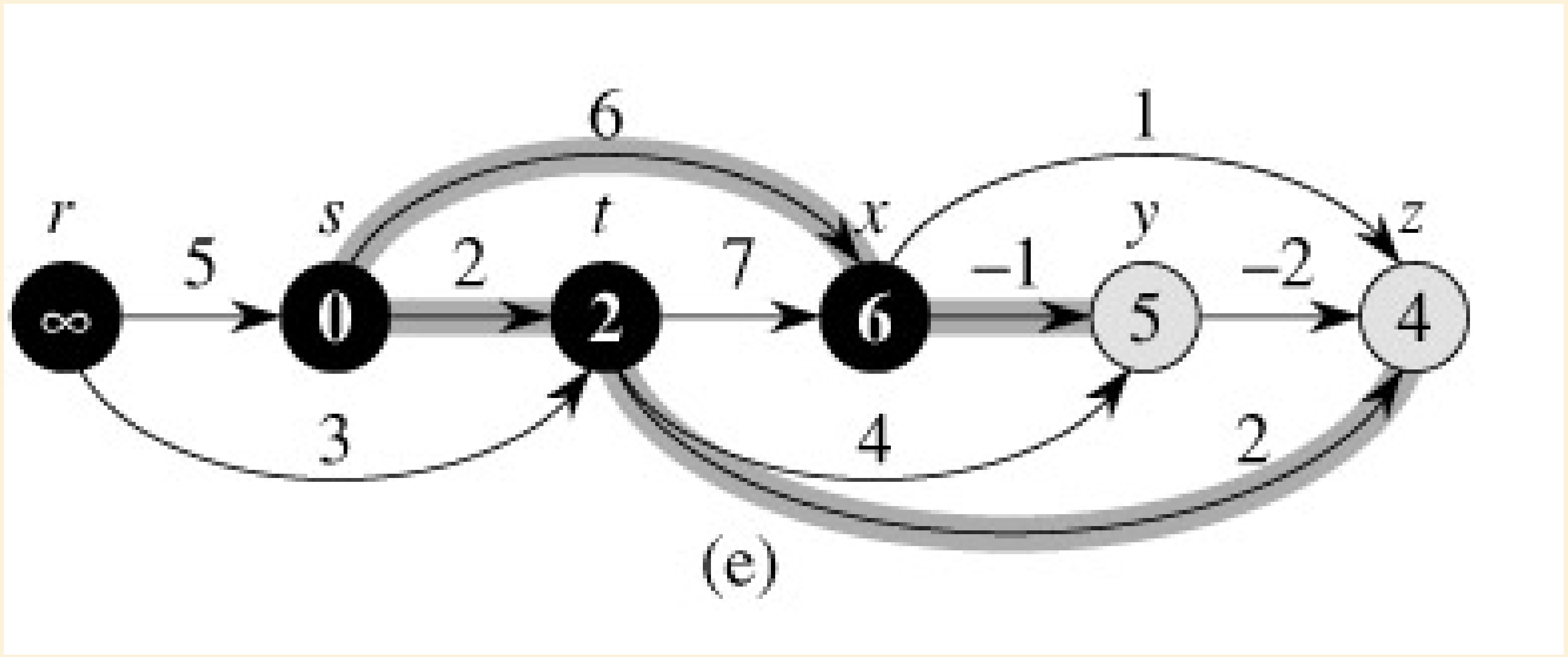
(c)

Example



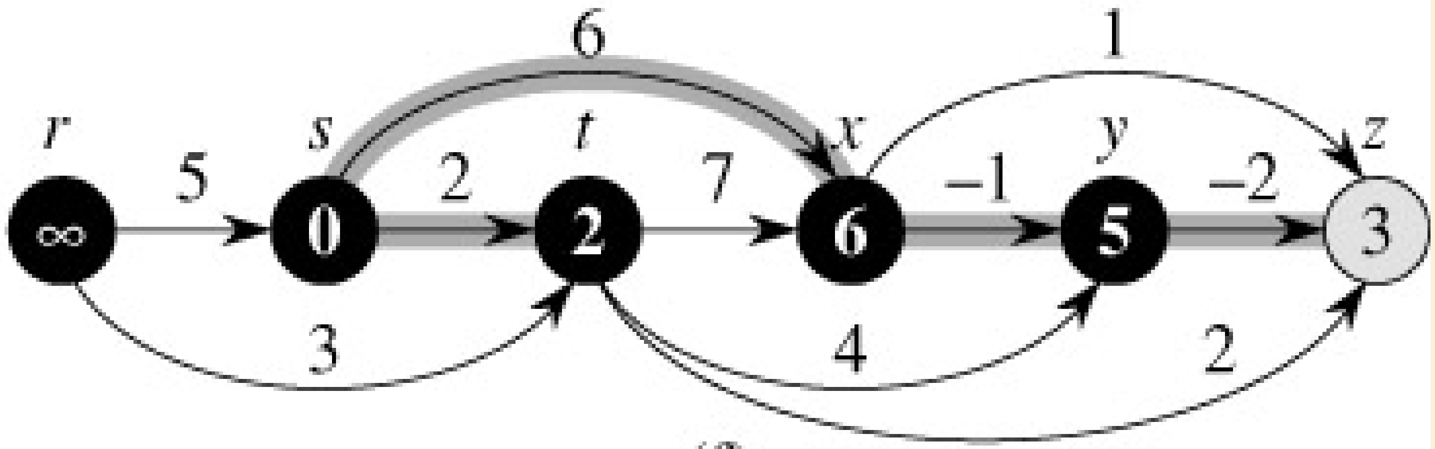
(d)

Example



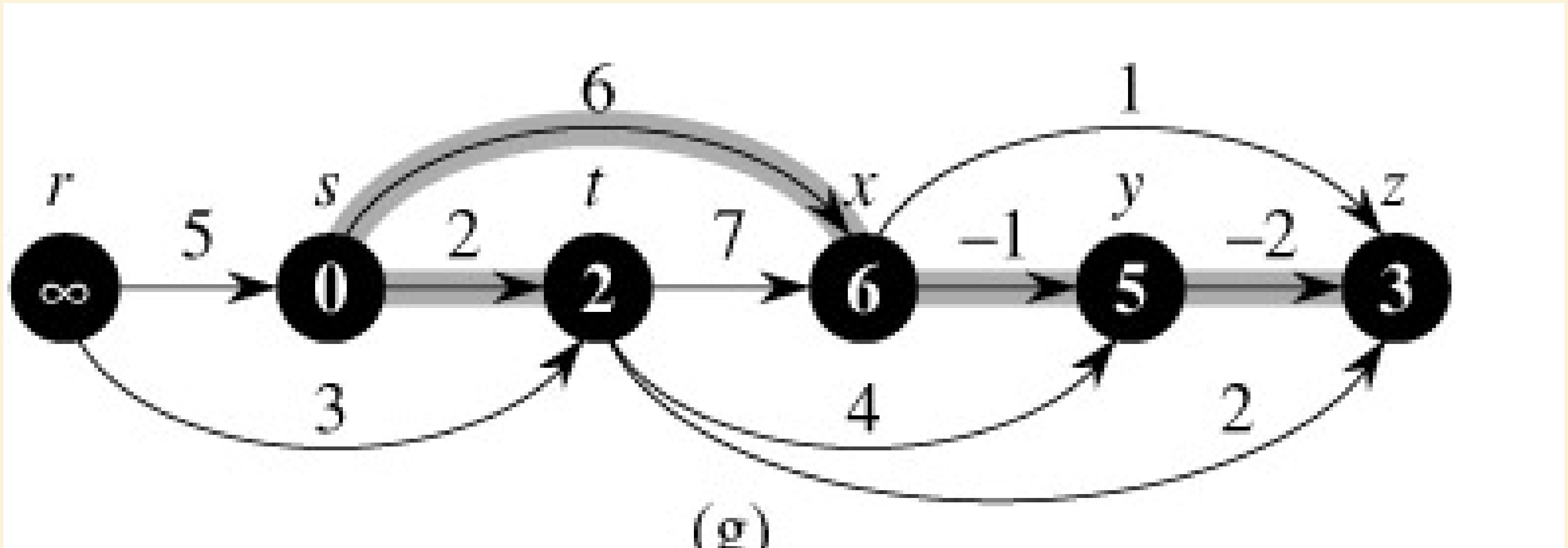
(e)

Example



(f)

Example



(g)

Correctness: Path relaxation property (Lemma 24.15)

Let $p = < v_0,\ v_1,\ .\ .\ .\ .\ ,\ v_k >$ be a shortest path from $s = v_0$ *to* v_k.

If we relax, in order, (v_0,v_1), (v_1,v_2), ... , (v_{k-1},v_k),

even intermixed with other relaxations,

then $d[v_k] = \delta(s,\ v_k)$.

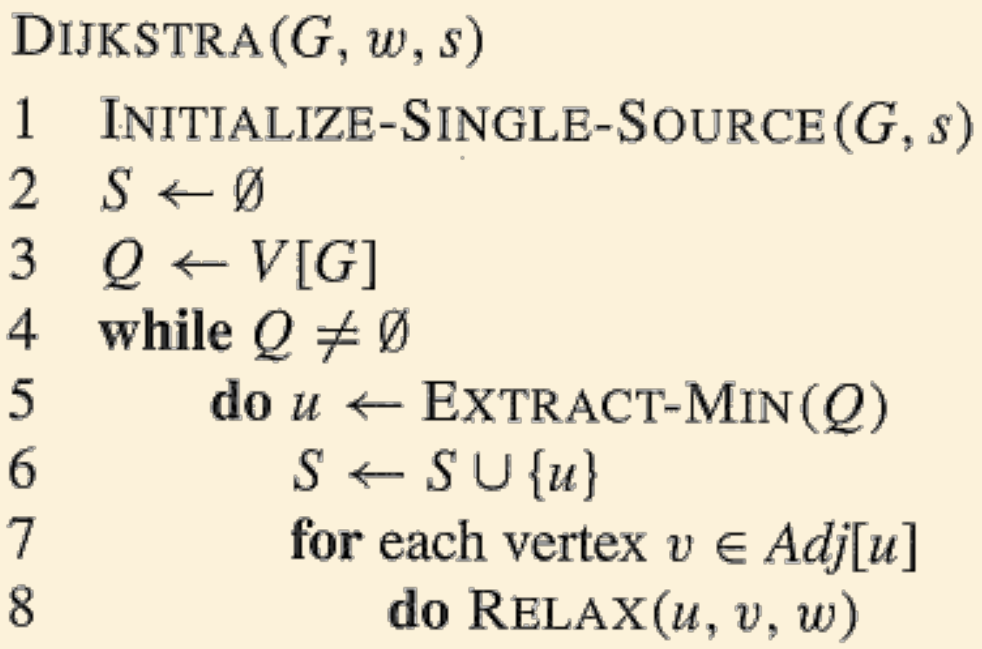
Correctness of DAG Shortest Path Algorithm

- Because we process vertices in topologically sorted order, edges of *any* path are relaxed in order of appearance in the path.
  - →Edges on any shortest path are relaxed in order.
  - →By path-relaxation property, correct.

Example: Dijkstra's algorithm

- Applies to general weighted directed graph (may contain cycles).
- But weights must be non-negative.
- Essentially a weighted version of BFS.
  - Instead of a FIFO queue, uses a priority queue.
  - Keys are shortest-path weights (d[v]).
- Maintain 2 sets of vertices:
  - S = vertices whose final shortest-path weights are determined.
  - Q = priority queue = V-S.

Dijkstra's algorithm

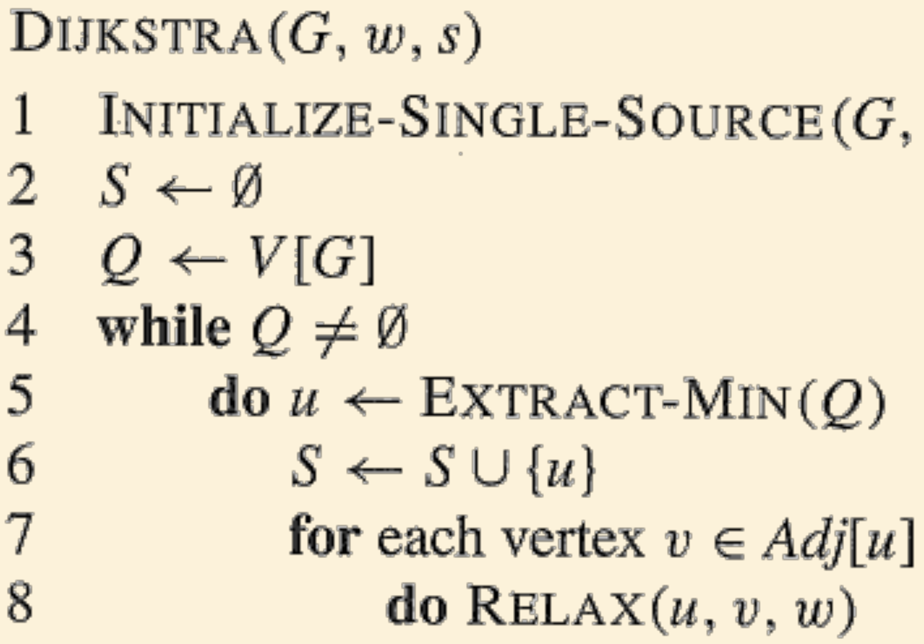


```
DIJKSTRA($G, w, s$)
INITIALIZE-SINGLE-SOURCE($G, s$)
$S \leftarrow \emptyset$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN($Q$)
       $S \leftarrow S \cup \{u\}$
       for each vertex $v \in Adj[u]$
           do RELAX($u, v, w$)
```

- Dijkstra's algorithm can be viewed as greedy, since it always chooses the "lightest" vertex in V – S to add to S.

Dijkstra's algorithm: *Analysis*

- *Analysis:*
  - Using minheap, queue operations takes *O(logV)* time



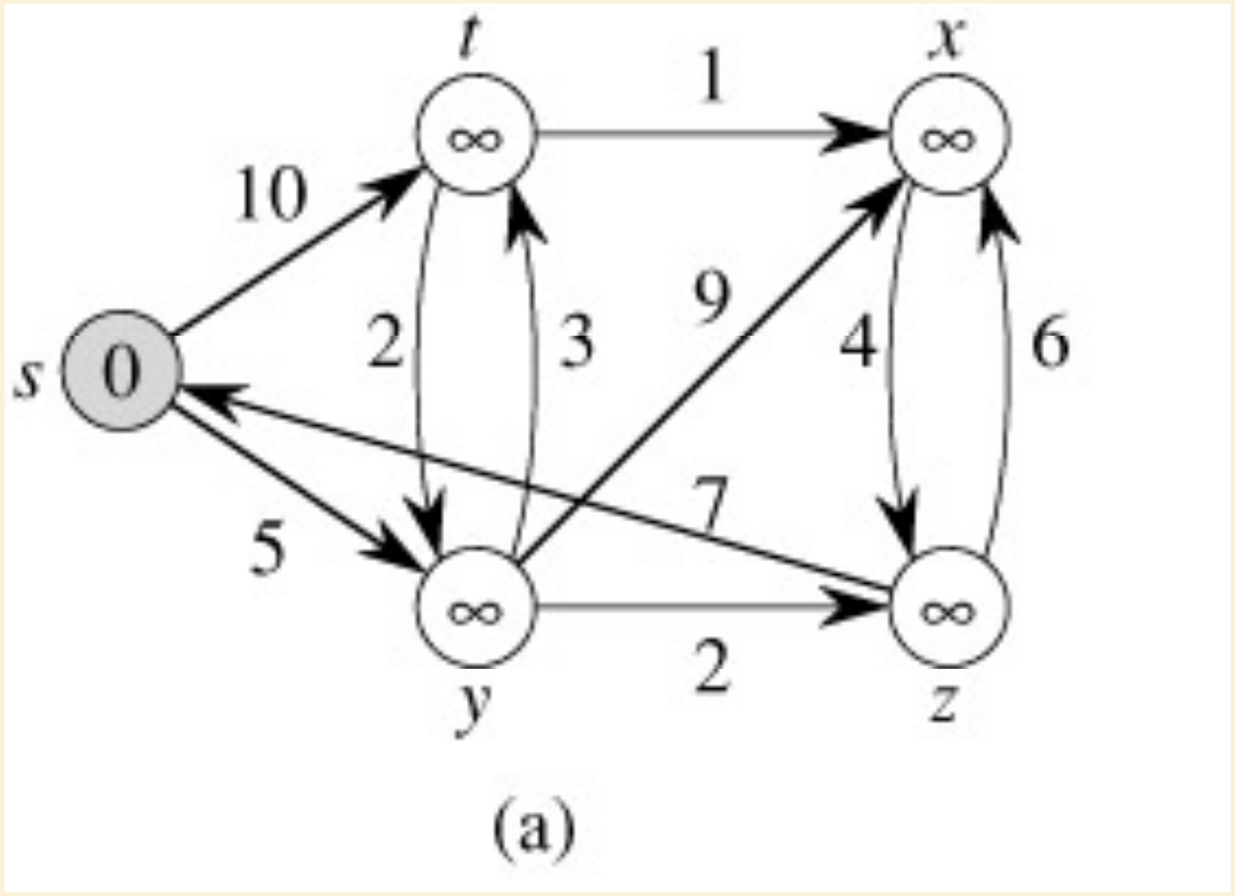
```
DIJKSTRA($G$, $w$, $s$)
1  INITIALIZE-SINGLE-SOURCE($G$, $s$)      $O(V)$
2  $S \leftarrow \emptyset$
3  $Q \leftarrow V[G]$
4  while $Q \neq \emptyset$
5      do $u \leftarrow$ EXTRACT-MIN($Q$)       $O(\log V) \times O(V)$ iterations
6         $S \leftarrow S \cup \{u\}$
7         for each vertex $v \in Adj[u]$
8             do RELAX($u$, $v$, $w$)       $O(\log V) \times O(E)$ iterations
```

$\rightarrow$ Running Time is $O(E \log V)$

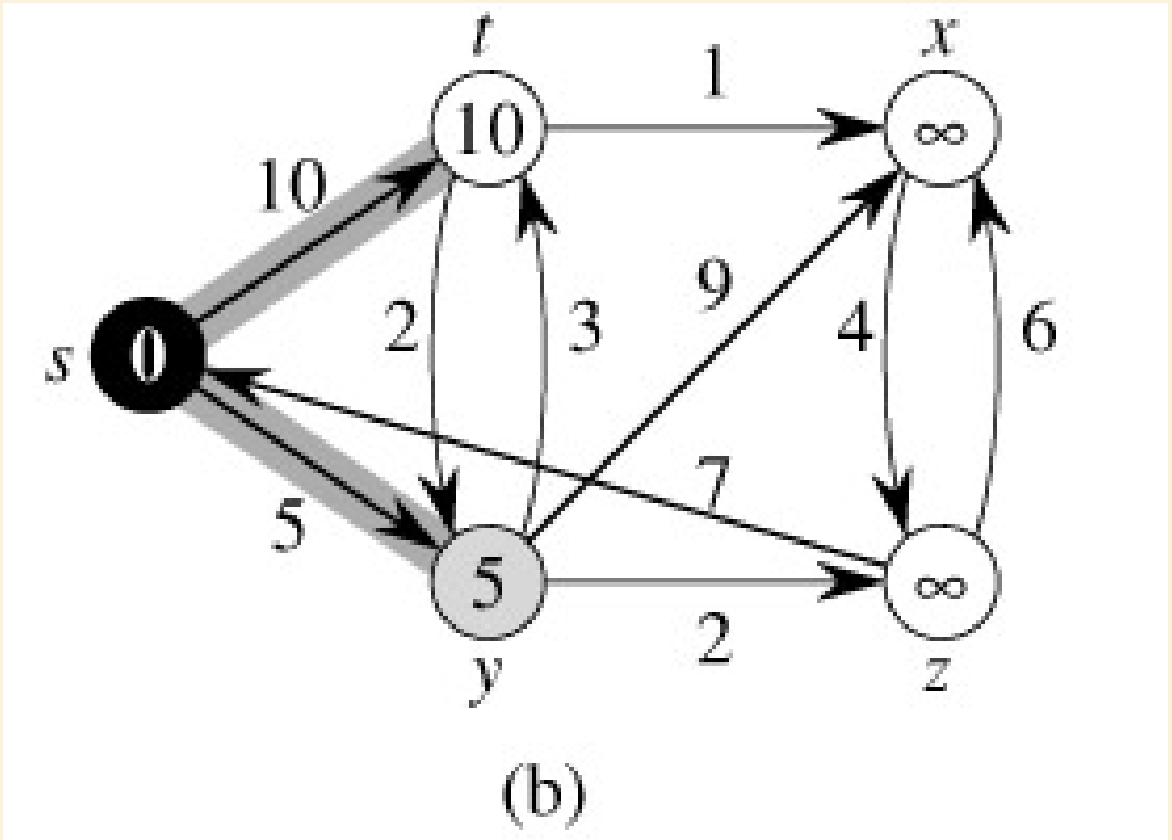
Example

Key:

White	⇔	Not Found
Grey	⇔	Handling
Black	⇔	Handled

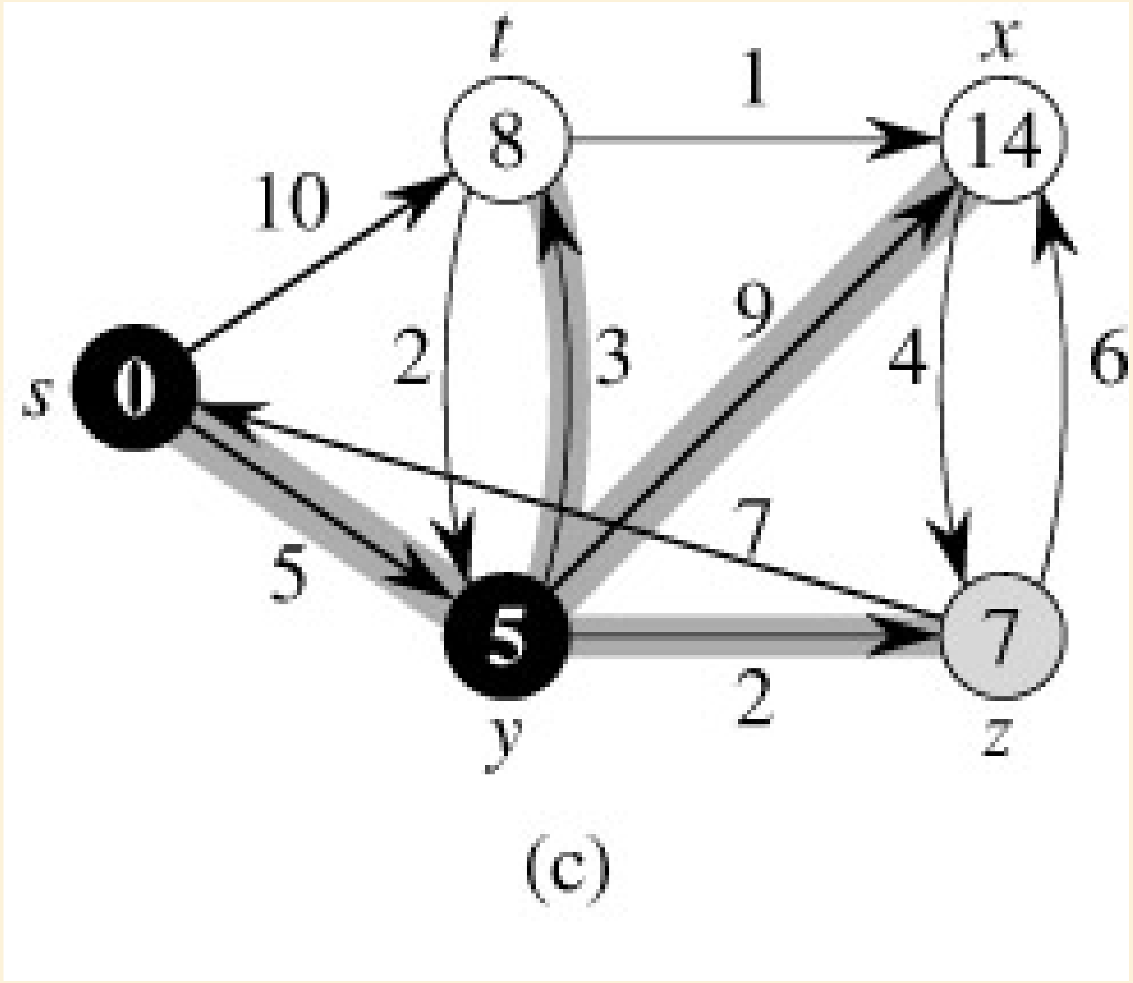


Example



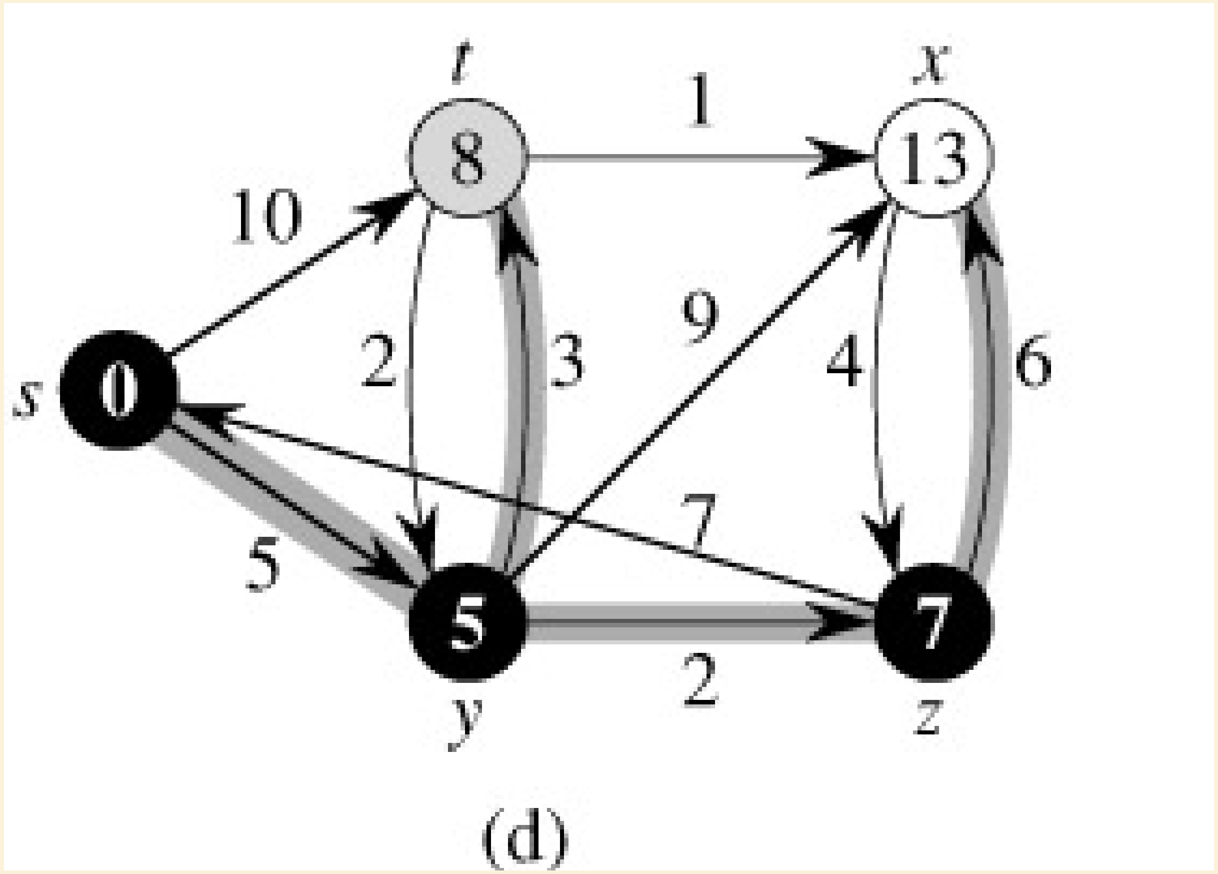
(b)

Example

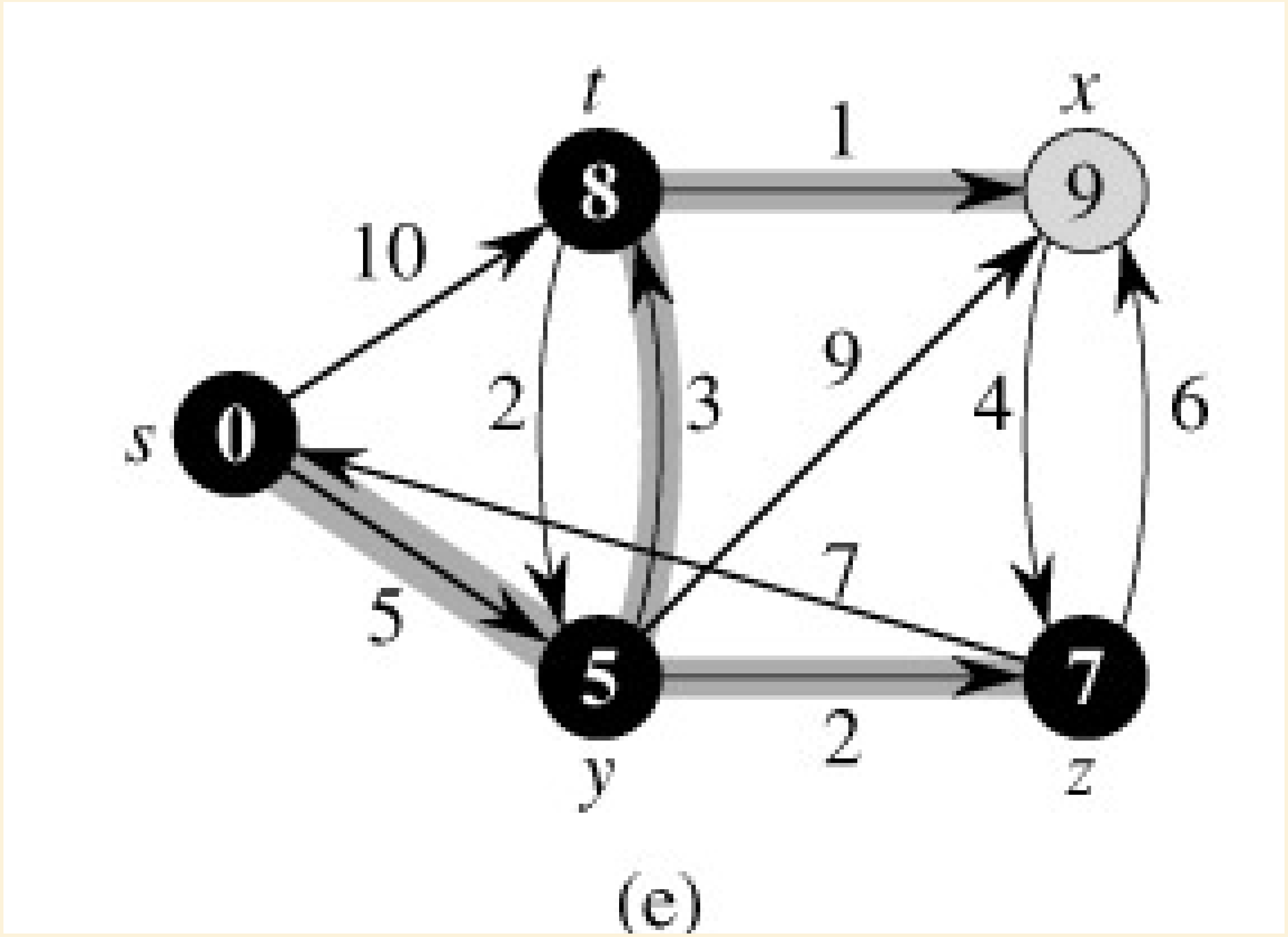


(c)

Example

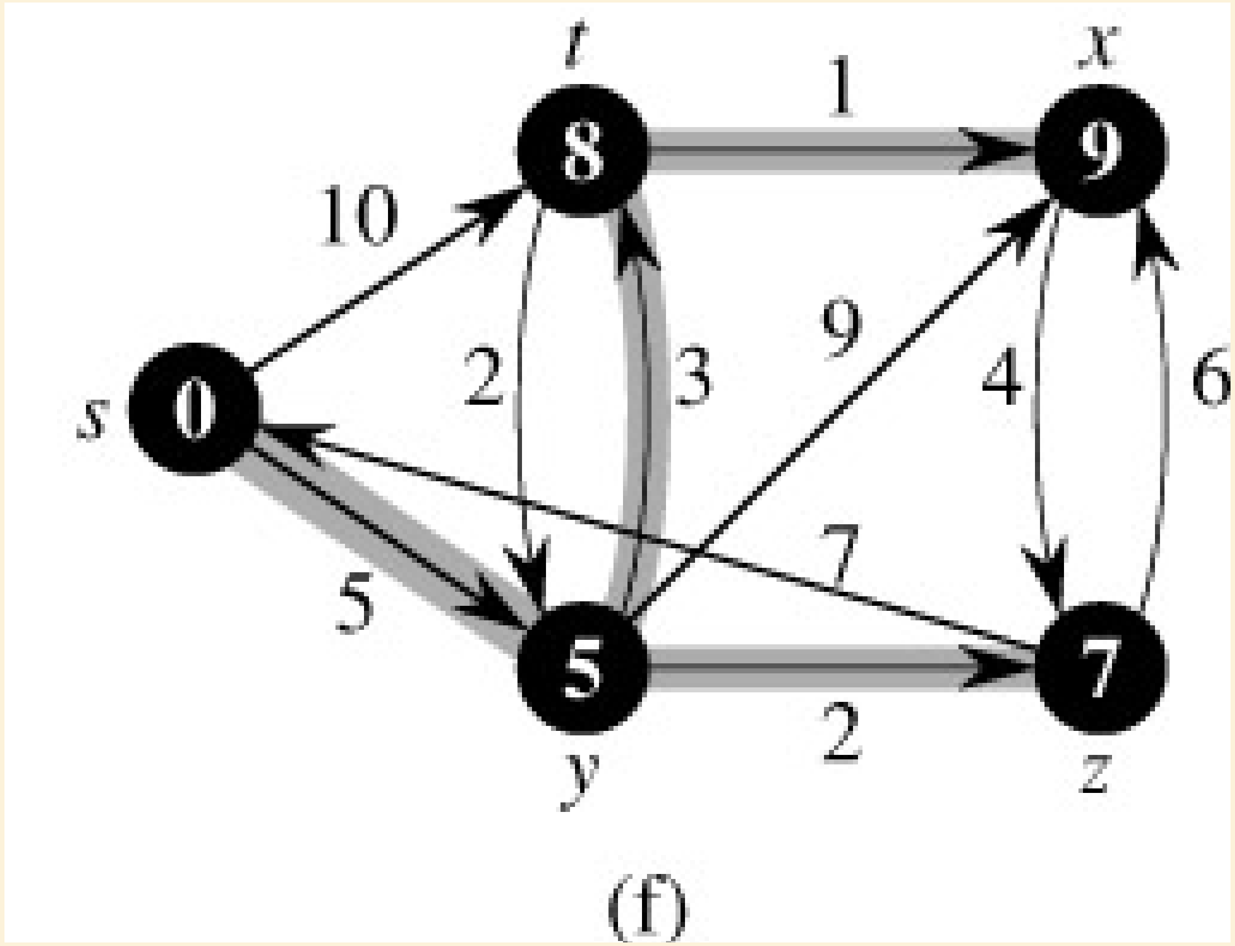


Example



(e)

Example



(f)

Correctness of Dijkstra's algorithm

```
DIJKSTRA($G, w, s$)
INITIALIZE-SINGLE-SOURCE($G, s$)
$S \leftarrow \emptyset$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN($Q$)
        $S \leftarrow S \cup \{u\}$
        for each vertex $v \in Adj[u]$
            do RELAX($u, v, w$)
```

- **Loop invariant:** d[v] = δ(s, v) for all v in S.
  - Initialization: Initially, S is empty, so trivially true.
  - Termination: At end, Q is empty → S = V → d[v] = δ(s, v) for all v in V.
  - Maintenance:
    - Need to show that
      - d[u] = δ(s, u) when u is added to S in each iteration.
      - d[u] does not change once u is added to S.

Correctness of Dijkstra's Algorithm: Upper Bound Property

- Upper Bound Property:

  1. $d[v] \geq \delta(s,v) \forall v \in V$
  2. Once $d[v] = \delta(s,v)$, it doesn't change

- Proof:

  By induction.

  Base Case: $d[v] \geq \delta(s,v) \forall v \in V$ immediately after initialization, since

  $d[s] = 0 = \delta(s,s)$

  $d[v] = \infty \forall v \neq s$

  Inductive Step:

  Suppose $d[x] \geq \delta(s,x) \forall x \in V$

  Suppose we relax edge (u,v).

  If $d[v]$ changes, then $d[v] = d[u] + w(u,v)$

  $$\geq \delta(s,u) + w(u,v)$$

  $$\geq \delta(s,v)$$

Correctness of Dijkstra's Algorithm

Claim: When u is added to S, $d[u] = \delta(s,u)$

Proof by Contradiction: Let u be the first vertex added to S such that $d[u] \neq \delta(s,u)$ when u is added.

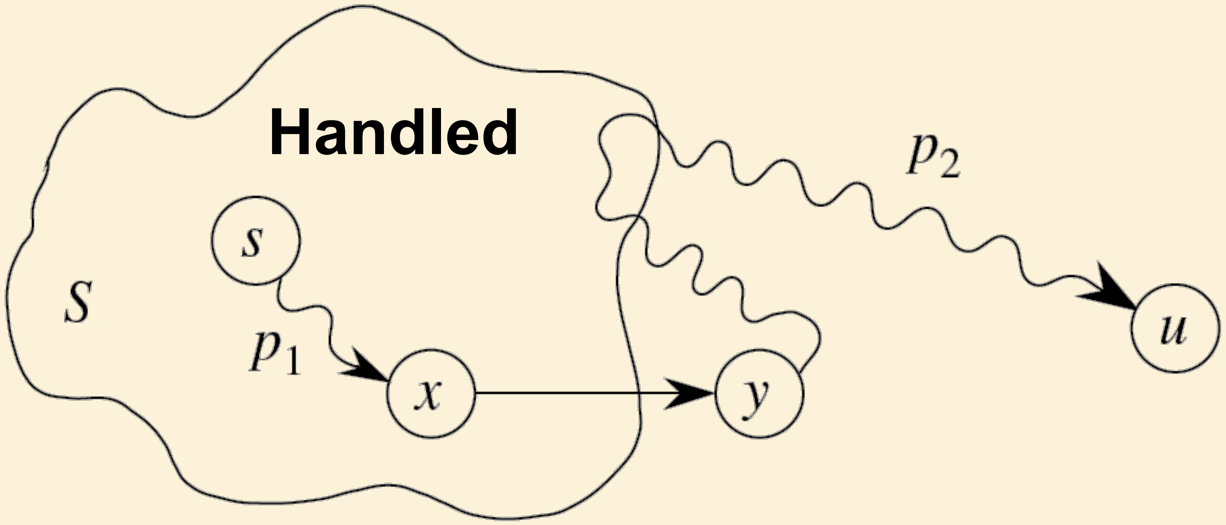
Let y be first vertex in $V - S$ on shortest path to u
Let x be the predecessor of y on the shortest path to u

Claim: $d[y] = \delta(s,y)$ when u is added to S.

Proof:

$d[x] = \delta(s,x)$, since $x \in S$.

(x,y) was relaxed when x was added to S $\rightarrow d[y] = \delta(s,x) + w(x,y) = \delta(s,y)$



Correctness of Dijkstra's Algorithm

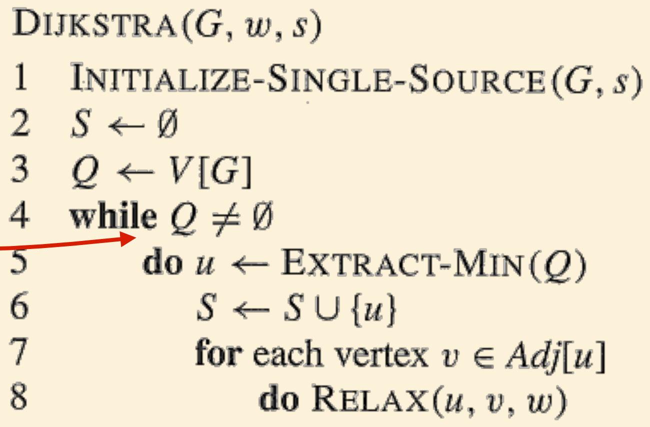
Thus $d[y] = \delta(s,y)$ when u is added to S.

$\rightarrow d[y] = \delta(s,y) \le \delta(s,u) \le d[u]$ (upper bound property)

But $d[u] \le d[y]$ when u added to S

Thus $d[y] = \delta(s,y) = \delta(s,u) = d[u]$!

Thus when u is added to S, $d[u] = \delta(s,u)$

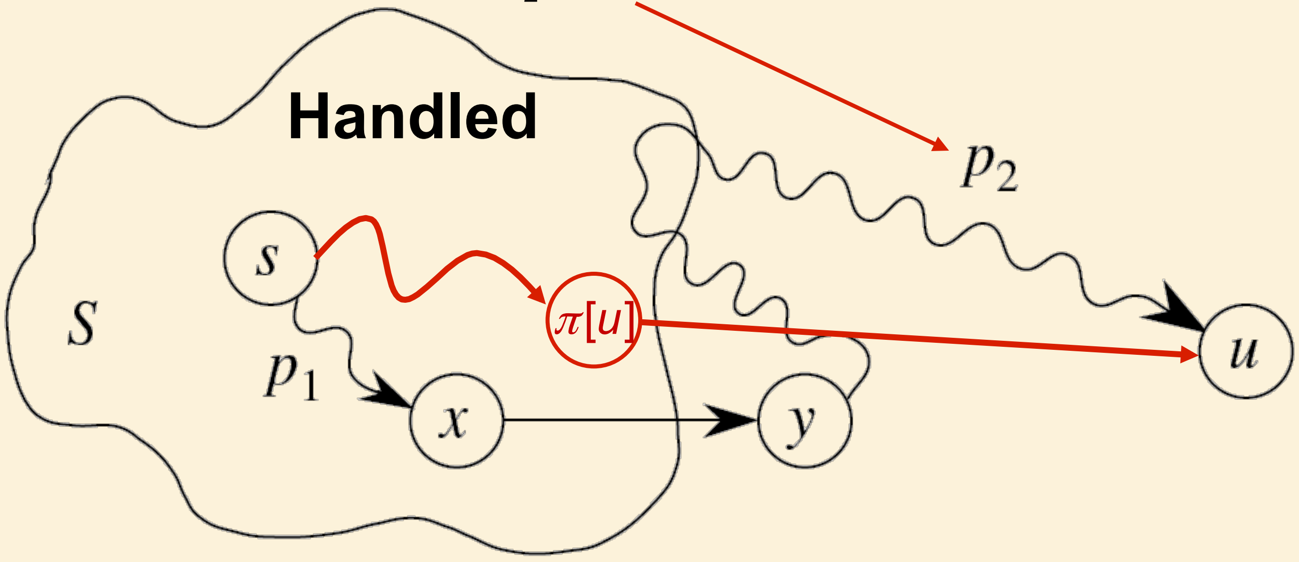


```
DIJKSTRA(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S ← ∅
3  Q ← V[G]
4  while Q ≠ ∅
5      do u ← EXTRACT-MIN(Q)
6         S ← S ∪ {u}
7         for each vertex v ∈ Adj[u]
8             do RELAX(u, v, w)
```

Consequences:

There is a shortest path to u such that the predecessor of u $\pi[u] \in S$ when u is added to S.

The path through y can only be a shortest path if $w[p_2] = 0$.



Correctness of Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S ← ∅
3  Q ← V[G]
4  while Q ≠ ∅
5        do u ← EXTRACT-MIN(Q)
6           S ← S ∪ {u}
7           for each vertex v ∈ Adj[u]
8               do RELAX(u, v, w)
```

Relax(u,v,w) can only decrease $d[v]$.

By the upper bound property, $d[v] \geq \delta(s,v)$.

Thus once $d[v] = \delta(s,v)$, it will not be changed.

- **Loop invariant:** d[v] = δ(s, v) for all v in S.
  - Maintenance:
    - Need to show that
      - d[u] = δ(s, u) when u is added to S in each iteration. ✓
      - d[u] does not change once u is added to S. ?