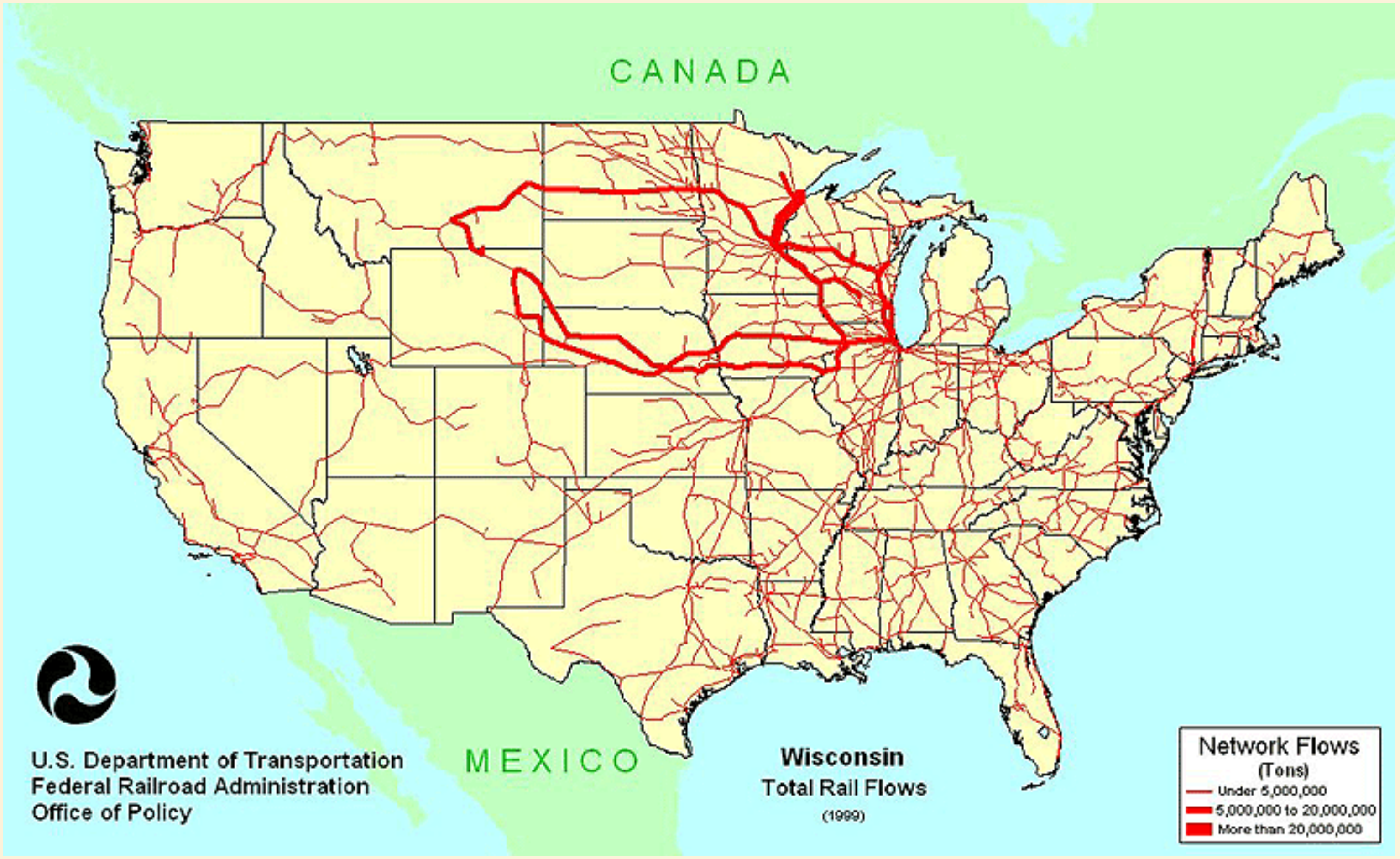
The Maximum Network Flow Problem

Network Flows



Types of Networks

- Internet
- Telephone
- Cell
- Highways
- Rail
- Electrical Power
- Water
- Sewer
- Gas
- …

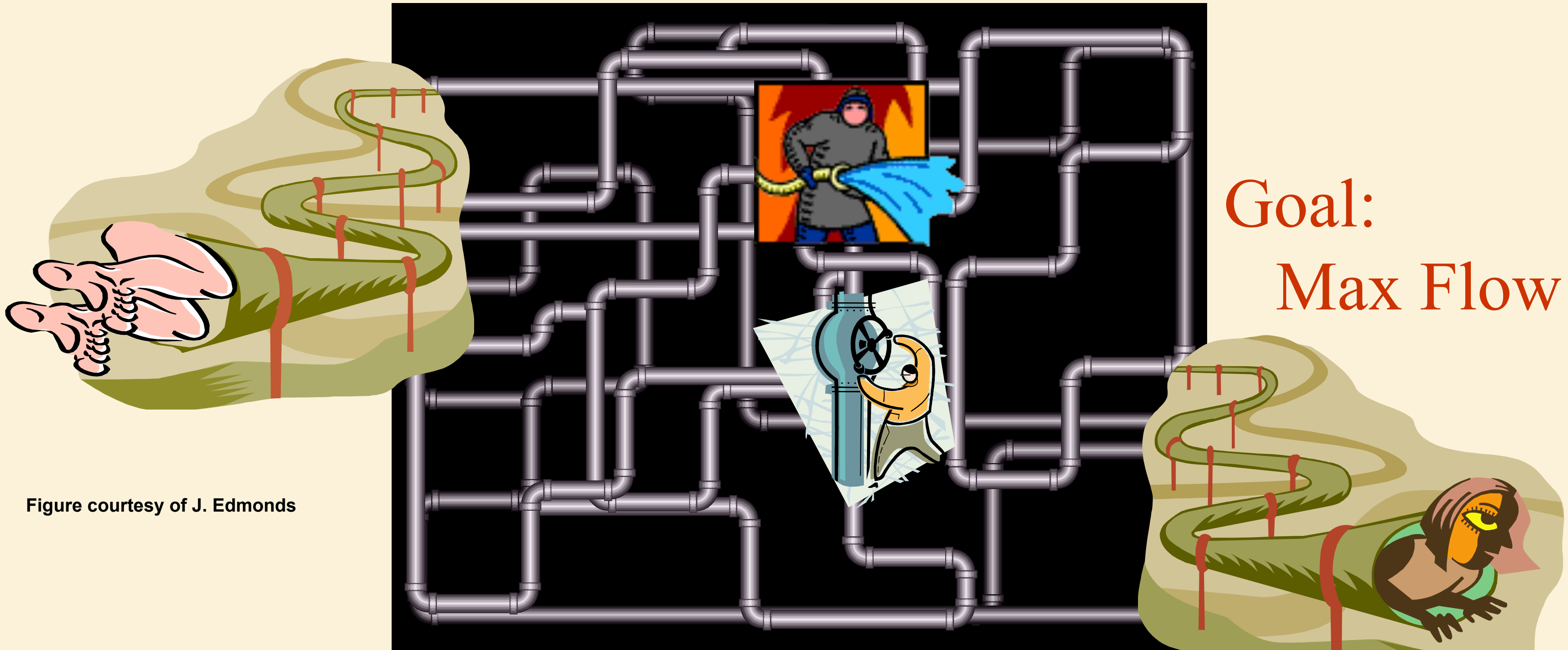
Maximum Flow Problem

- How can we maximize the flow in a network from a source or set of sources to a destination or set of destinations?
- The problem reportedly rose to prominence in relation to the rail networks of the Soviet Union, during the 1950's. The US wanted to know how quickly the Soviet Union could get supplies through its rail network to its satellite states in Eastern Europe.
- In addition, the US wanted to know which rails it could destroy most easily to cut off the satellite states from the rest of the Soviet Union. It turned out that these two problems were closely related, and that solving the **max flow problem** also solves the **min cut problem** of figuring out the cheapest way to cut off the Soviet Union from its satellites.

Source: lbackstrom, The Importance of Algorithms, at www.topcoder.com

Network Flow

- Instance:
  - A Network is a directed graph G
  - Edges represent pipes that carry flow
  - Each edge (u,v) has a maximum capacity c(u,v)
  - A source node s in which flow arrives
  - A sink node t out which flow leaves

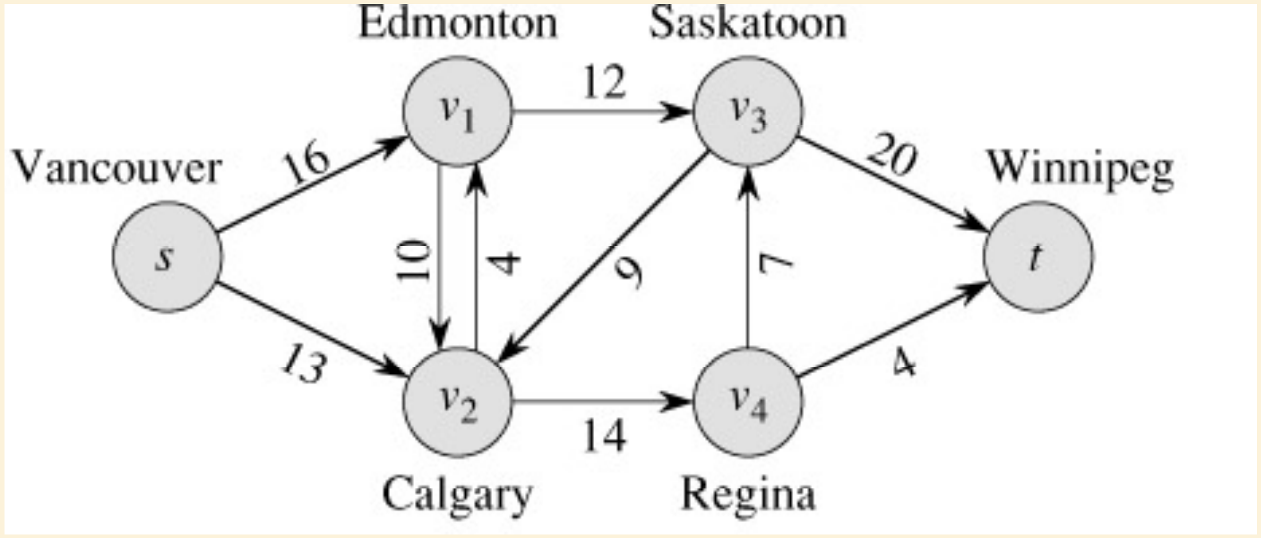


Goal:
Max Flow

Figure courtesy of J. Edmonds

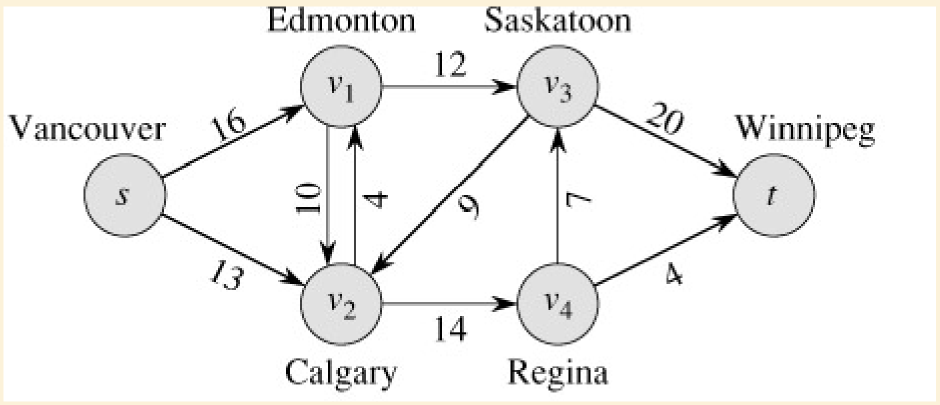
The Problem

- Use a graph to model material that flows through conduits.
- Each edge represents one conduit, and has a capacity, which is an upper bound on the flow rate, in units/time.
- Can think of edges as pipes of different sizes.
- Want to compute max rate that we can ship material from a designated source to a designated sink.



What is a Flow Network?

- Each edge (u,v) has a nonnegative capacity c(u,v).
- If (u,v) is not in E, assume c(u,v)=0.
- We have a source s, and a sink t.
- Assume that every vertex v in V is on some path from s to t.
- e.g., $c(s,v_1)=16$; $c(v_1,s)=0$; $c(v_2,v_3)=0$



What is a Flow in a Network?

- For each edge (u,v), the flow f(u,v) is a real-valued function that must satisfy 3 conditions:

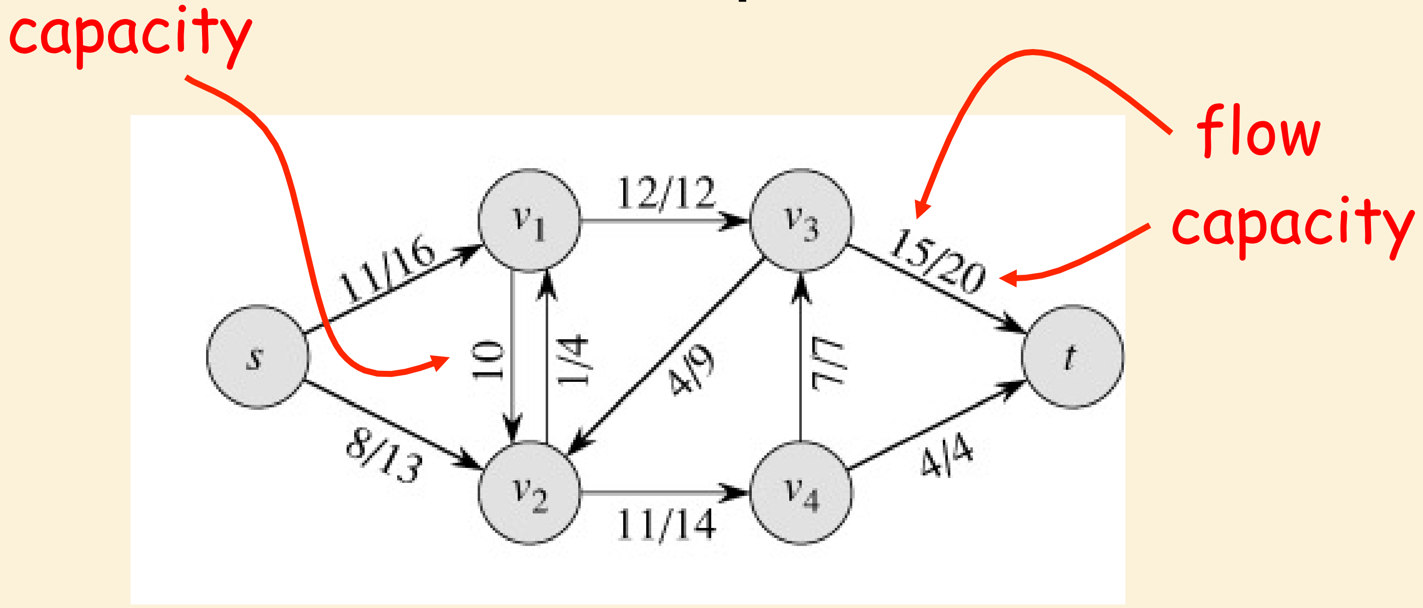
Capacity constraint: $\forall u,v \in V,\ f(u,v) \le c(u,v)$

Skew symmetry: $\forall u,v \in V,\ f(u,v) = -f(v,u)$

Flow conservation: $\forall u \in V - \{s,t\},\ \sum_{v \in V} f(u,v) = 0$

- Notes:
  - The skew symmetry condition implies that f(u,u)=0.
  - We show only the *positive* flows in the flow network.

Example of a Flow:



- $f(v_2, v_1) = 1$, $c(v_2, v_1) = 4$.
- $f(v_1, v_2) = -1$, $c(v_1, v_2) = 10$.
- $f(v_3, s) + f(v_3, v_1) + f(v_3, v_2) + f(v_3, v_4) + f(v_3, t) =$

  $0 + (-12) + 4 + (-7) + 15 = 0$

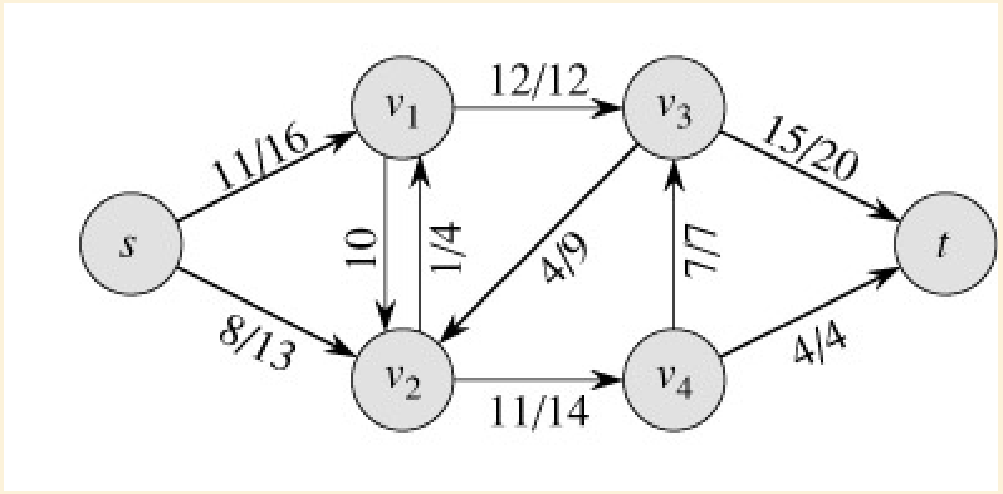
The Value of a flow

- The value of a flow is given by

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t)$$

- This is the total flow leaving s = the total flow arriving in t.

Example:



$|f| = f(s, v_1) + f(s, v_2) + f(s, v_3) + f(s, v_4) + f(s, t) =$

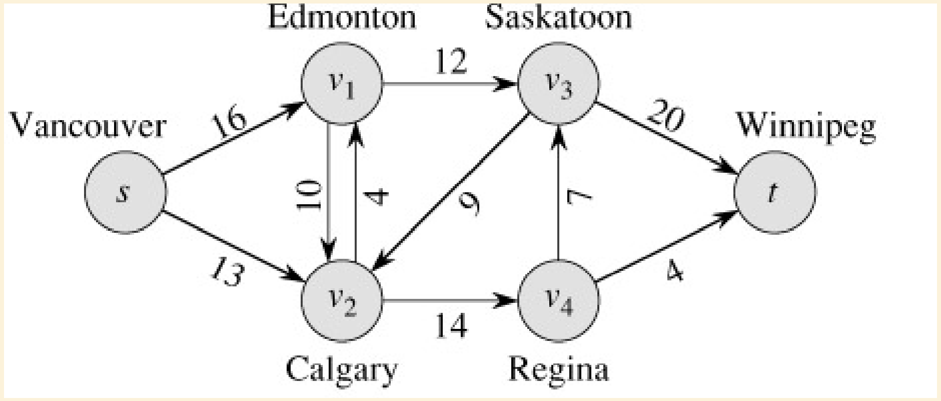
$11 + 8 + 0 + 0 + 0 = 19$

$|f| = f(s, t) + f(v_1, t) + f(v_2, t) + f(v_3, t) + f(v_4, t) =$

$0 + 0 + 0 + 15 + 4 = 19$

A flow in a network

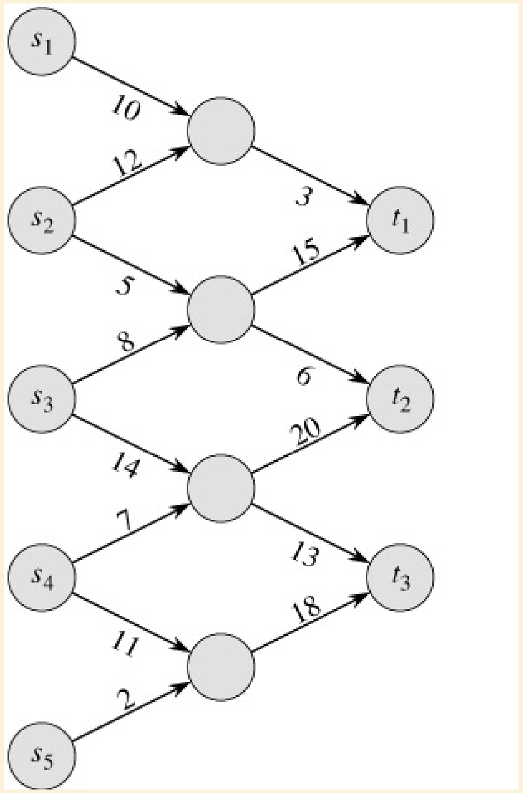
- We assume that there is only flow in one direction at a time.

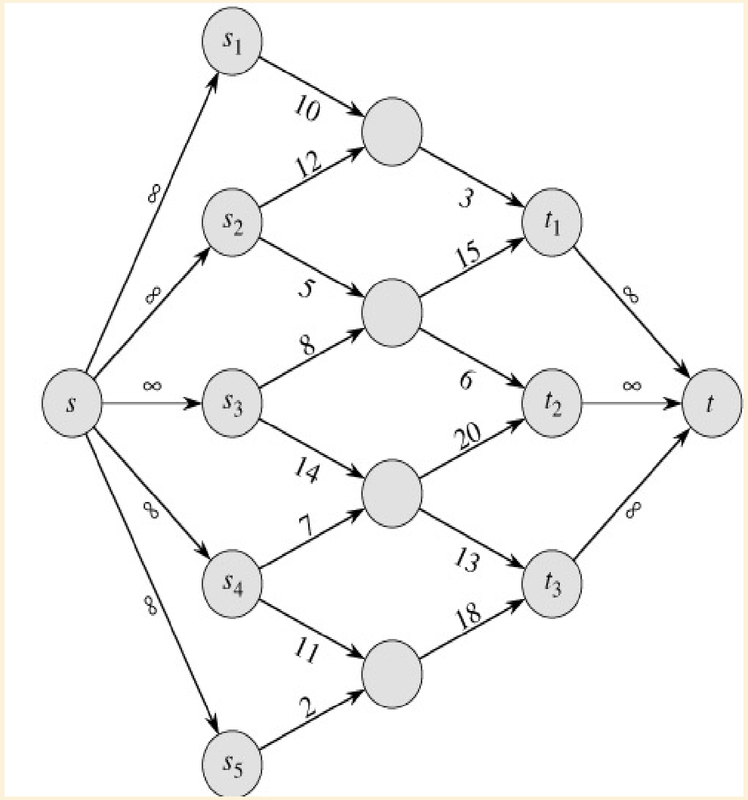


- Sending 7 trucks from Edmonton to Calgary and 3 trucks from Calgary to Edmonton has the same net effect as sending 4 trucks from Edmonton to Calgary.

Multiple Sources Network

- We have several sources and several targets.
- Want to maximize the total flow from all sources to all targets.
- Reduce to max-flow by creating a supersource and a supersink:





Residual Networks

- The residual capacity of an edge *(u,v)* in a network with a flow *f* is given by:

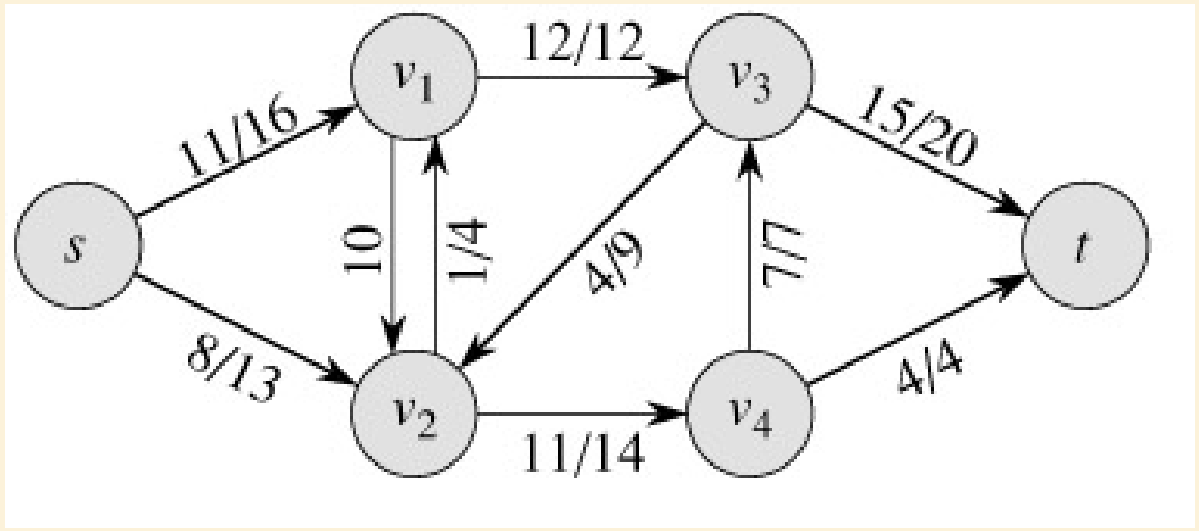
$$\boxed{c_f(u,v) = c(u,v) - f(u,v)}$$

- The residual network of a graph *G* induced by a flow *f* is the graph including only the edges with positive residual capacity, i.e.,

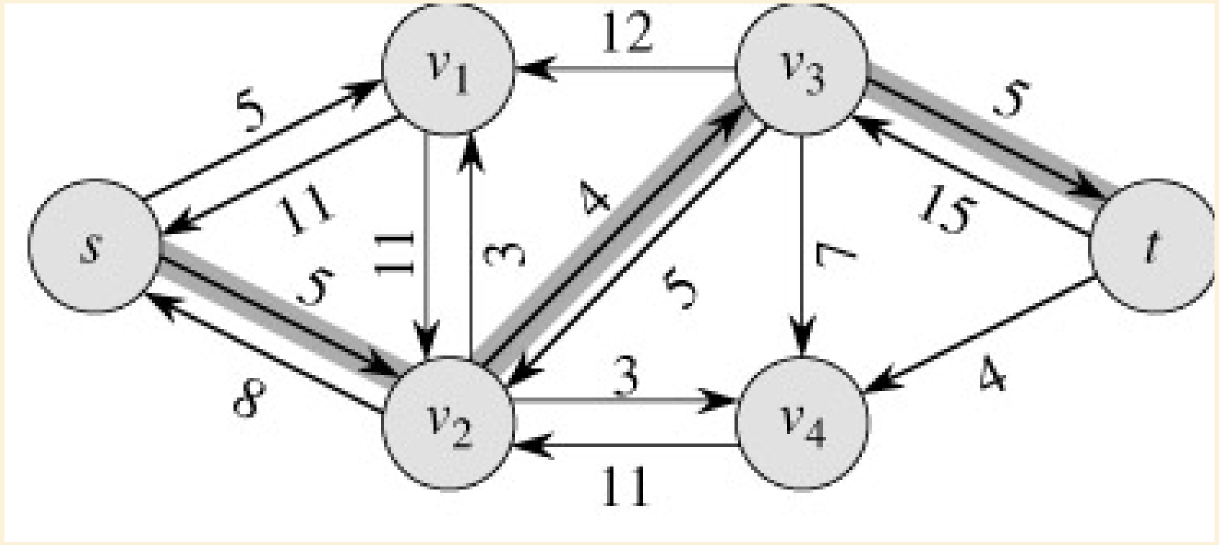
$G_f = (V, E_f)$, where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$

Example of Residual Network

Flow Network:



Residual Network:



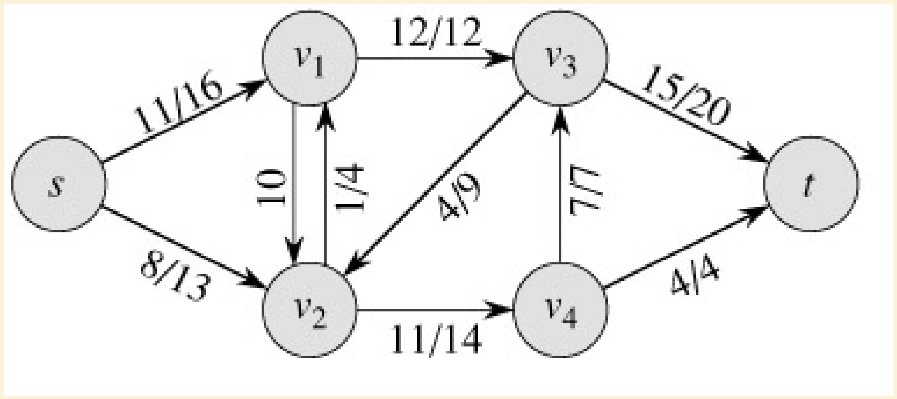
Augmenting Paths

- An augmenting path *p* is a simple path from s to t on the residual network.
- We can put more flow from *s* to *t* through *p*.
- We call the maximum capacity by which we can increase the flow on *p* the residual capacity of *p*.

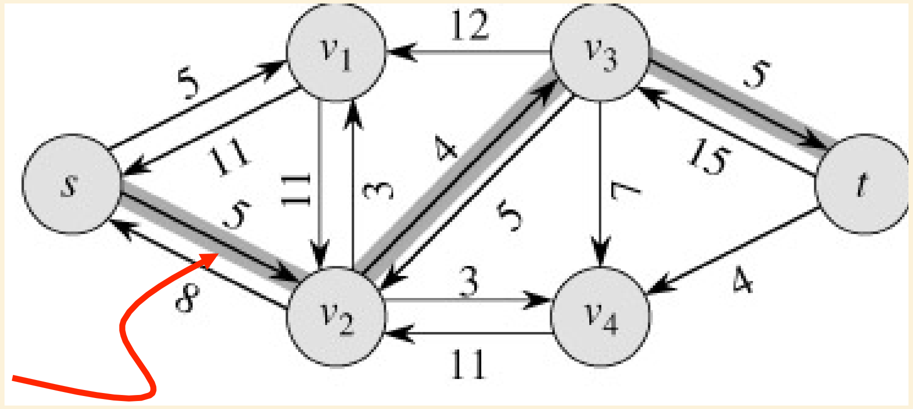
$$\boxed{c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}}$$

Augmenting Paths

Network:



Residual Network:



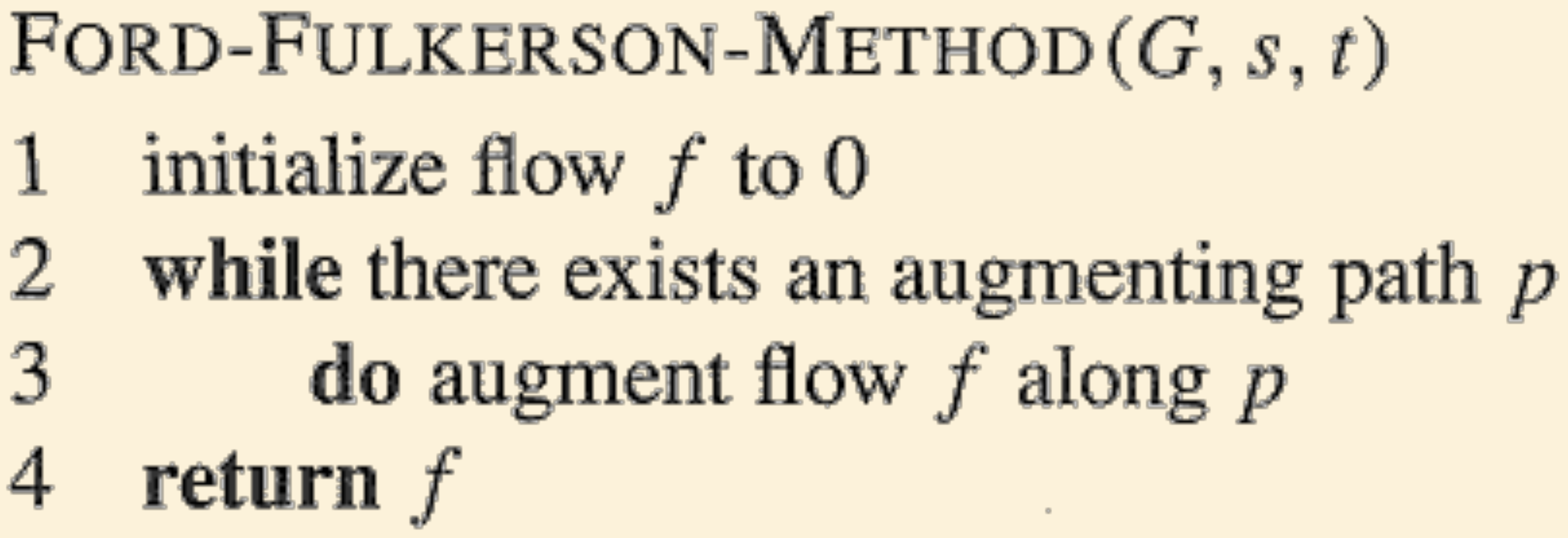
Augmenting path

The residual capacity of this augmenting path is 4.

Computing Max Flow

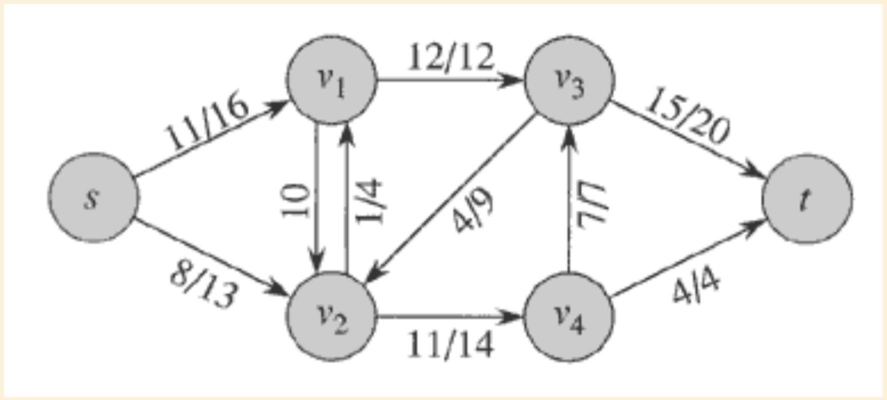
- Classic Method:
  - Identify augmenting path
  - Increase flow along that path
  - Repeat

Ford-Fulkerson Method

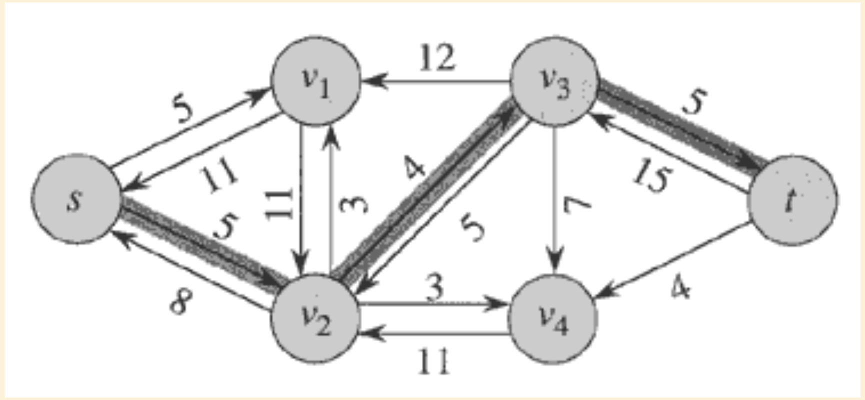


```
FORD-FULKERSON-METHOD($G, s, t$)
1  initialize flow $f$ to 0
2  while there exists an augmenting path $p$
3        do augment flow $f$ along $p$
4  return $f$
```

Example

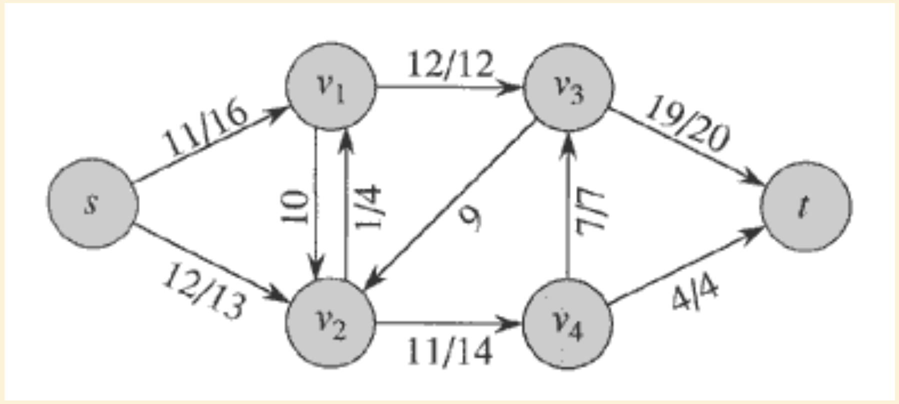


Flow(1)

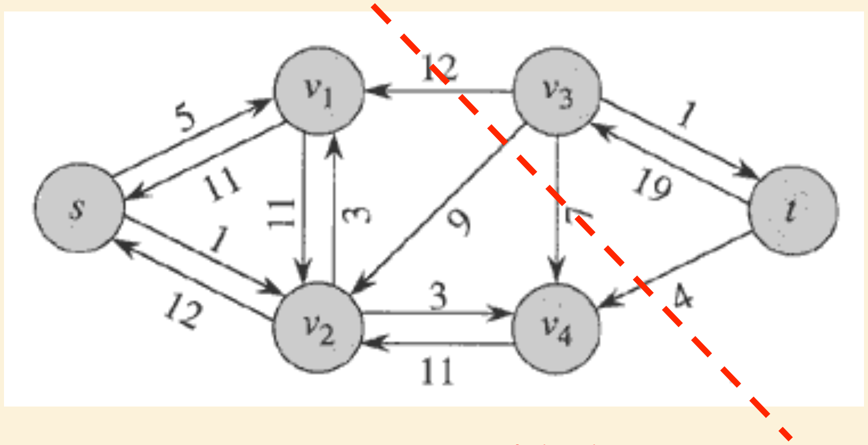


Residual(1)

No more augmenting paths → max flow attained.



Flow(2)

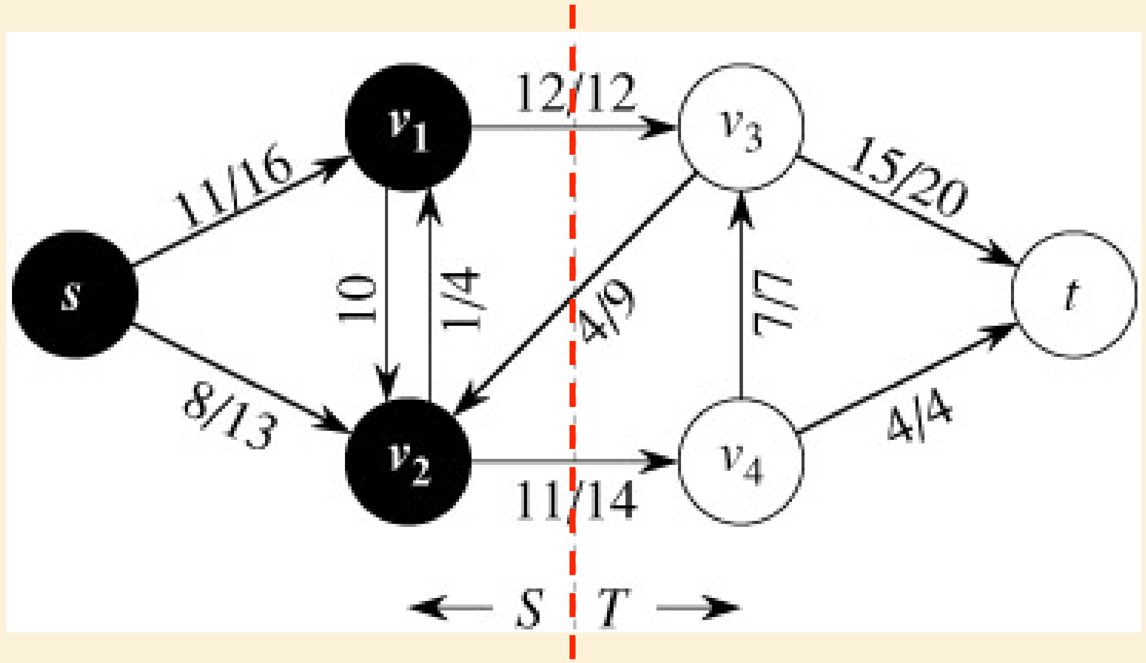


Residual(2)

Cut

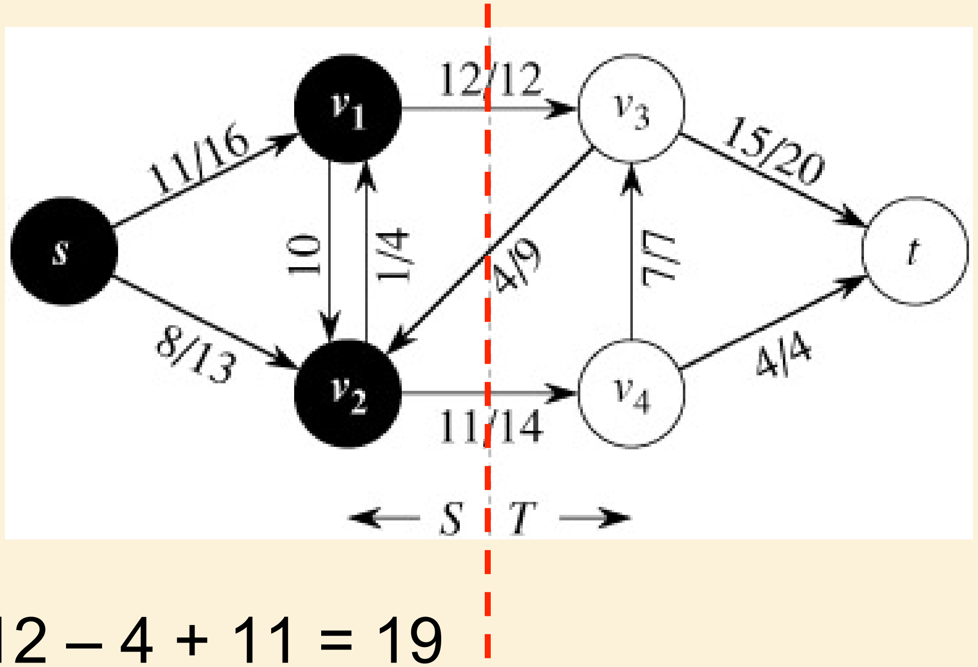
Cuts of Flow Networks

A cut (S,T) of a flow network is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.



The Net Flow through a Cut (S,T)

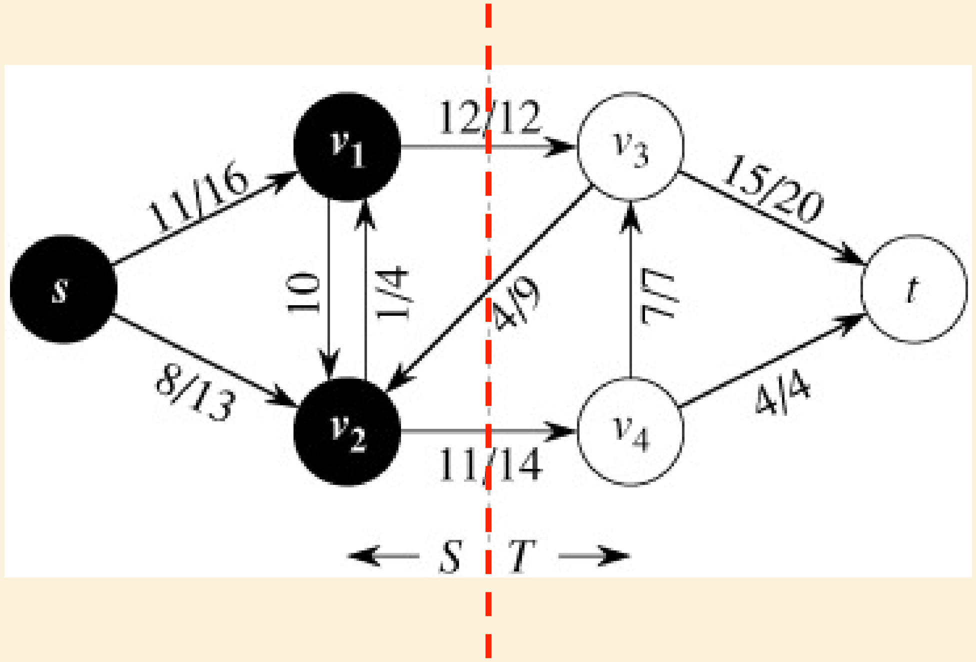
$$f(S,T) = \sum_{u \in S, v \in T} f(u,v)$$



- f(S,T) = 12 – 4 + 11 = 19

The Capacity of a Cut (S,T)

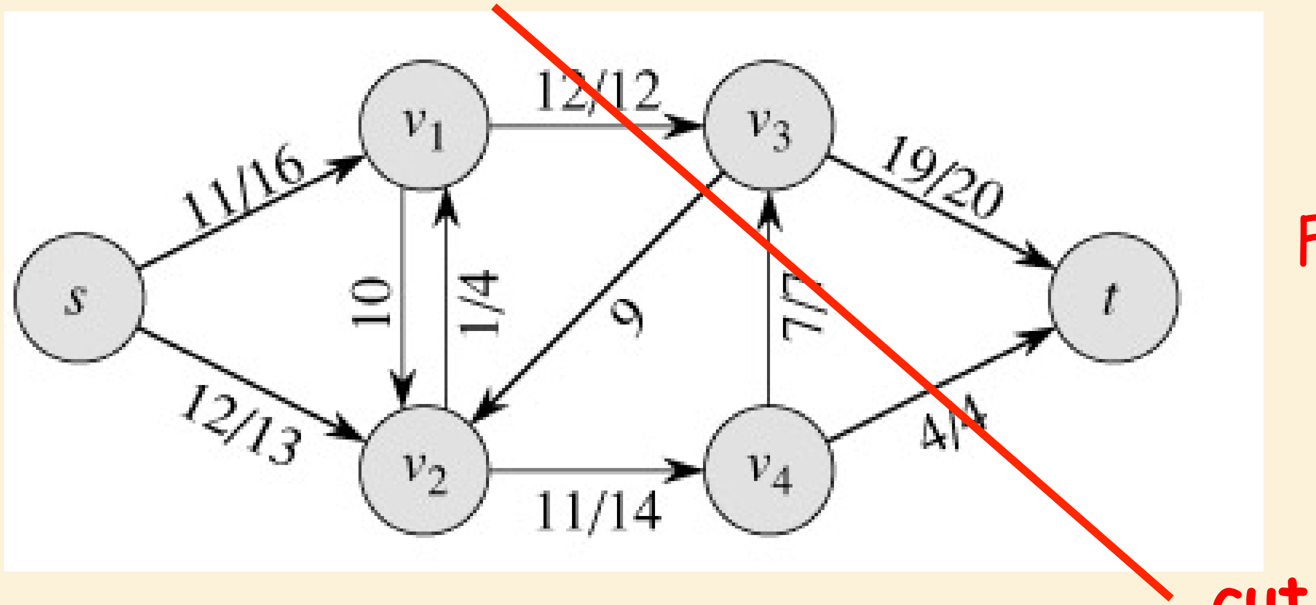
$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$



- c(S,T)= 12+ 0 + 14 = 26

Augmenting Paths – example

- **Capacity of the cut**
  = maximum possible flow through the cut
  = 12 + 7 + 4 = 23

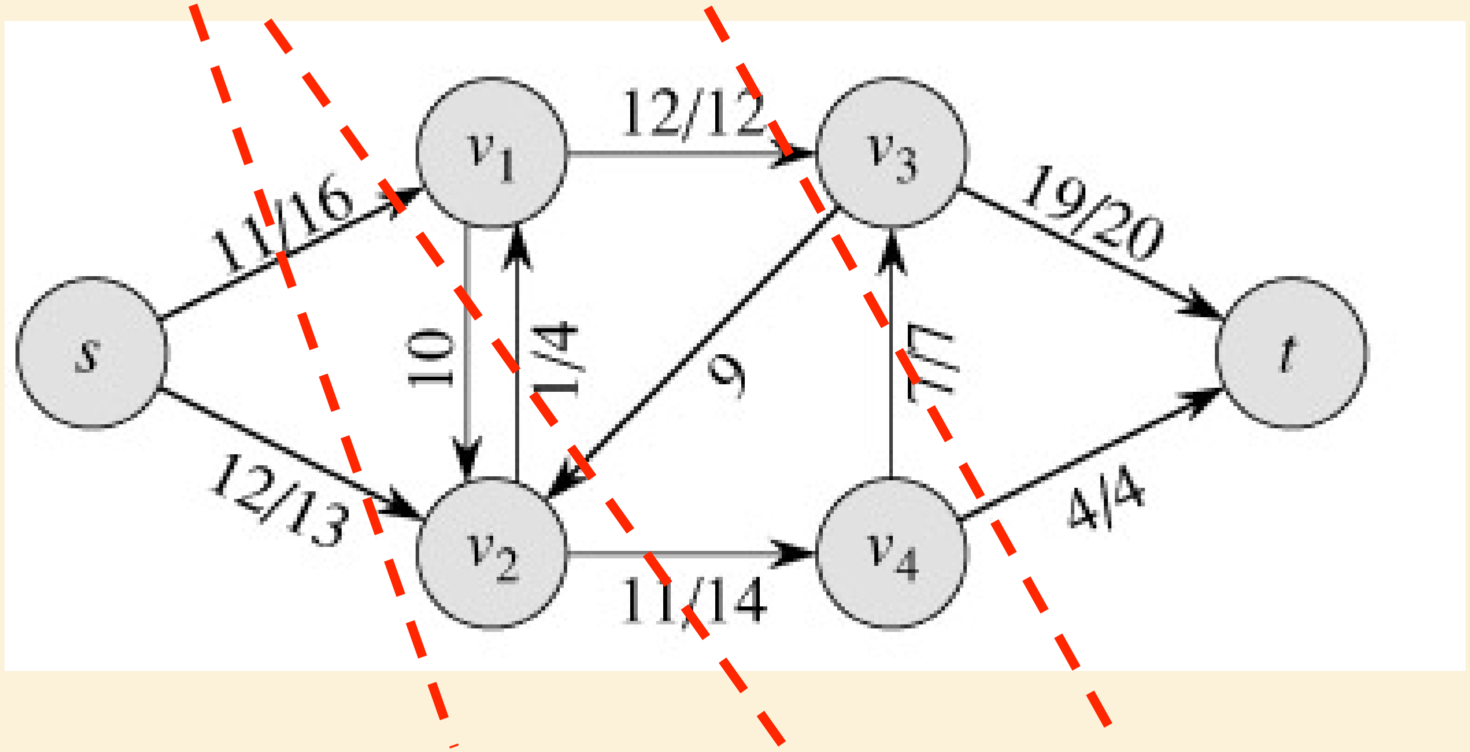


Flow(2)

- The network has a capacity of **at most** 23.
- In this case, the network **does** have a capacity of 23, because this is a **minimum cut**.

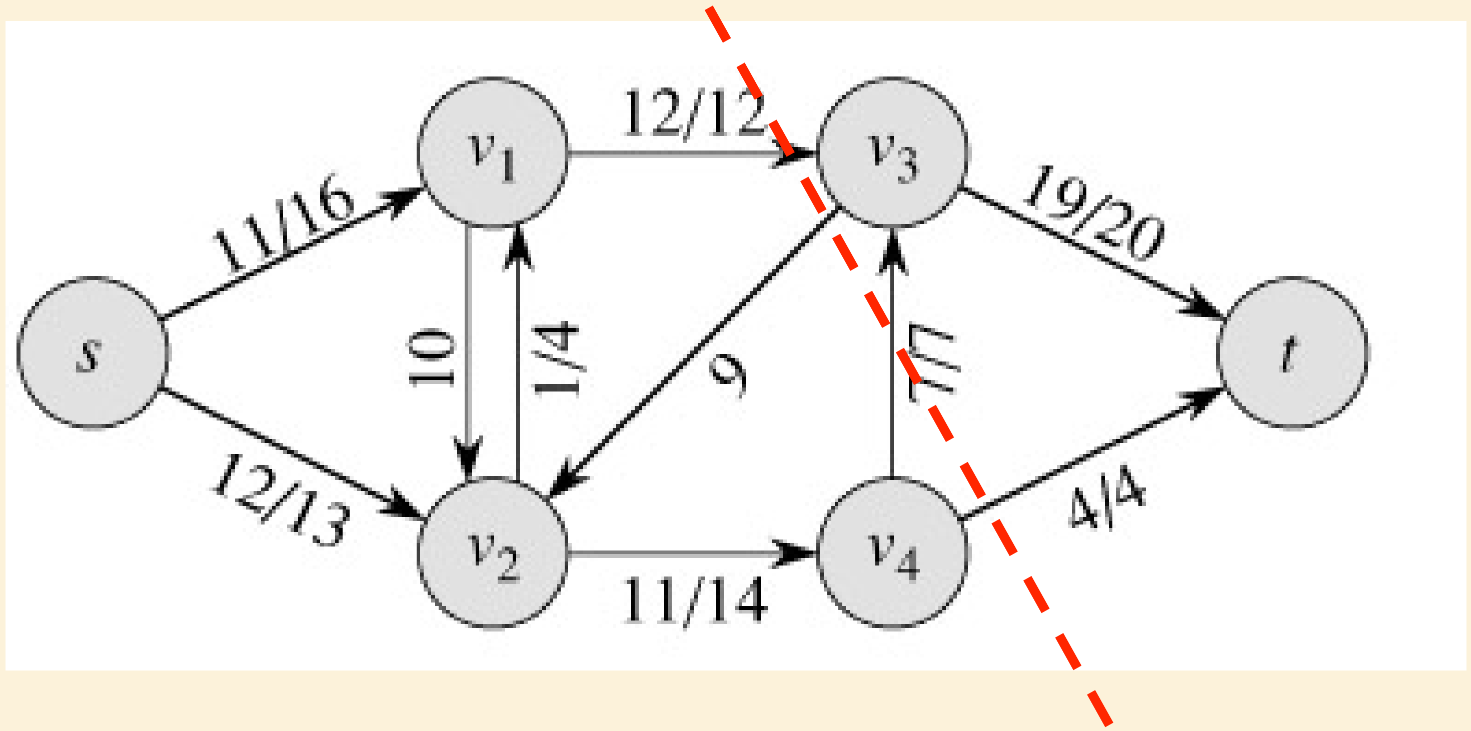
Net Flow of a Network

- The net flow across any cut is the same and equal to the flow of the network |f|.



Bounding the Network Flow

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.



Max-Flow Min-Cut Theorem

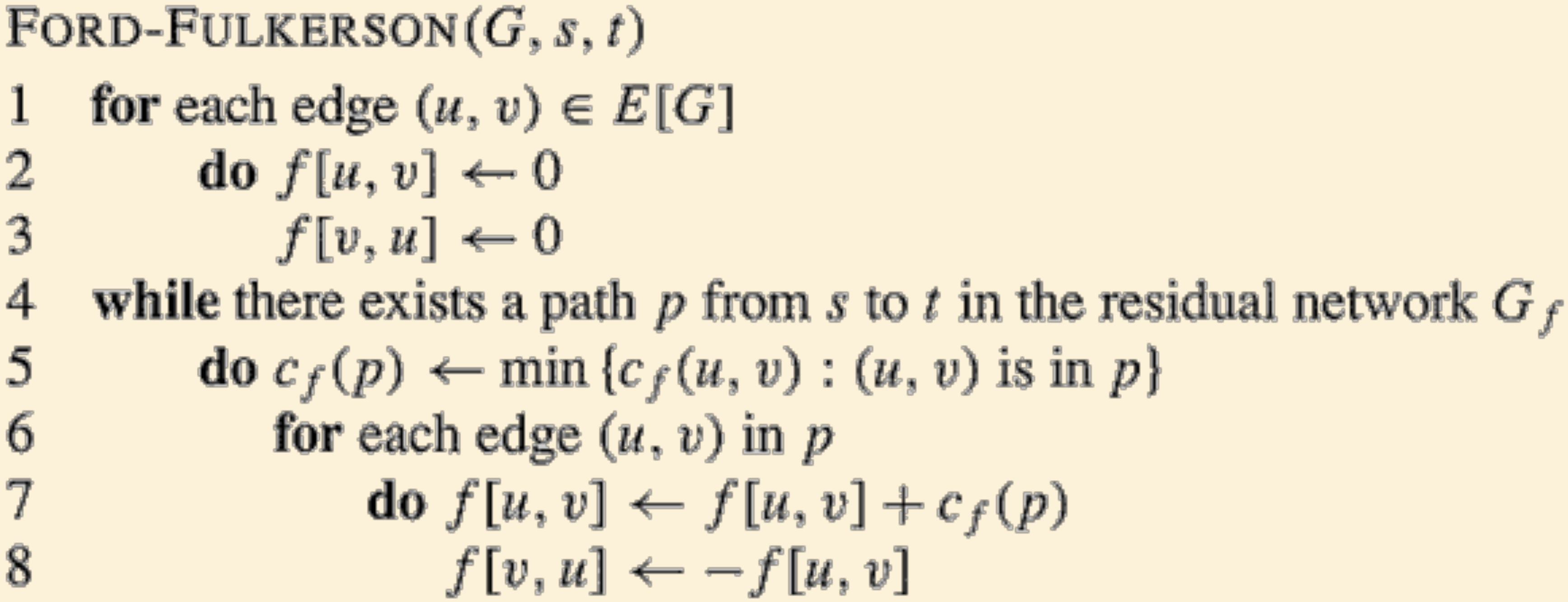
- If f is a flow in a flow network $G=(V,E)$, with source s and sink t, then the following conditions are equivalent:

  1. f is a maximum flow in G.

  2. The residual network G_f contains no augmented paths.

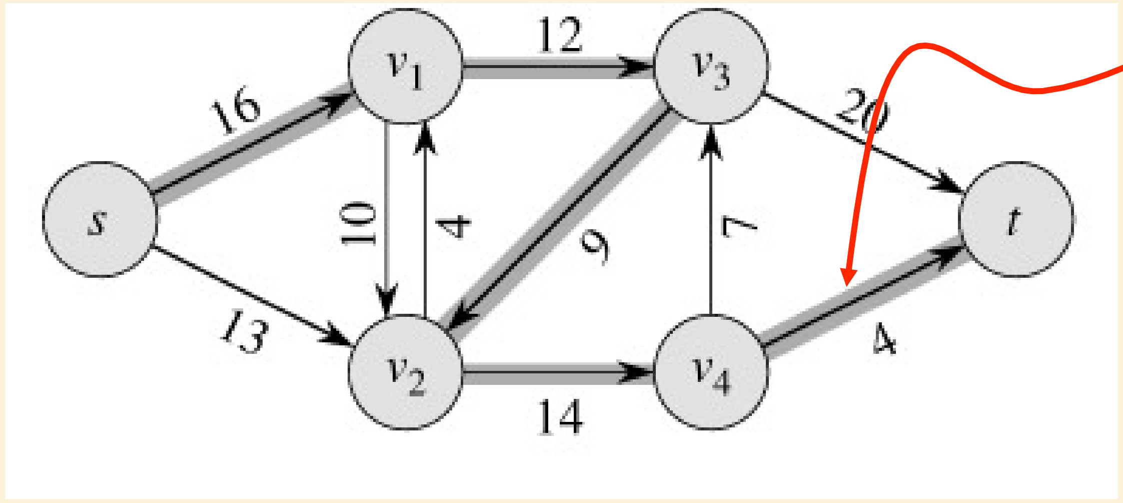
  3. $|f| = c(S,T)$ for some cut (S,T) (a min-cut).

The Basic Ford-Fulkerson Algorithm



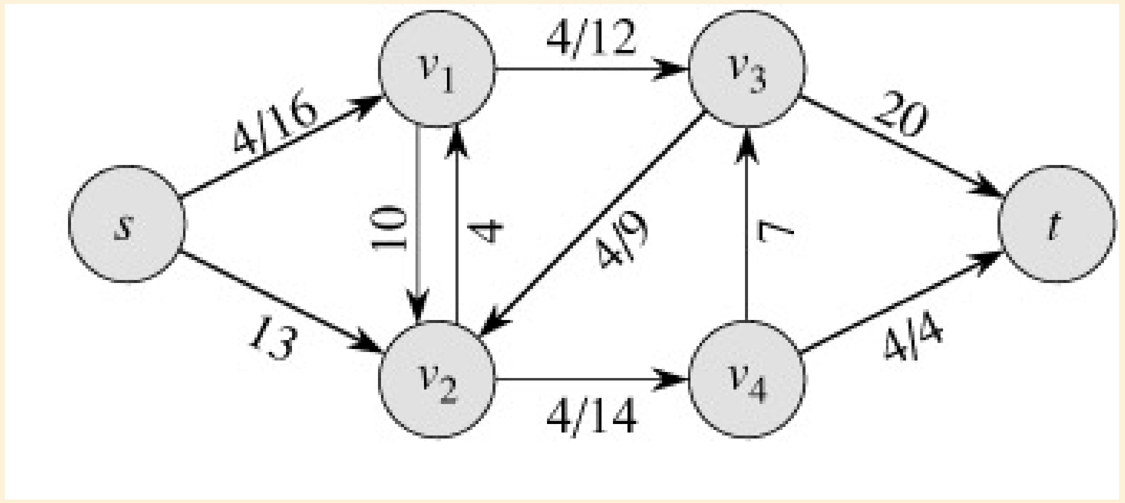
```
FORD-FULKERSON$(G, s, t)$
for each edge $(u, v) \in E[G]$
    do $f[u, v] \leftarrow 0$
       $f[v, u] \leftarrow 0$
while there exists a path $p$ from $s$ to $t$ in the residual network $G_f$
    do $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
       for each edge $(u, v)$ in $p$
           do $f[u, v] \leftarrow f[u, v] + c_f(p)$
              $f[v, u] \leftarrow -f[u, v]$
```

Example



augmenting path

Original Network

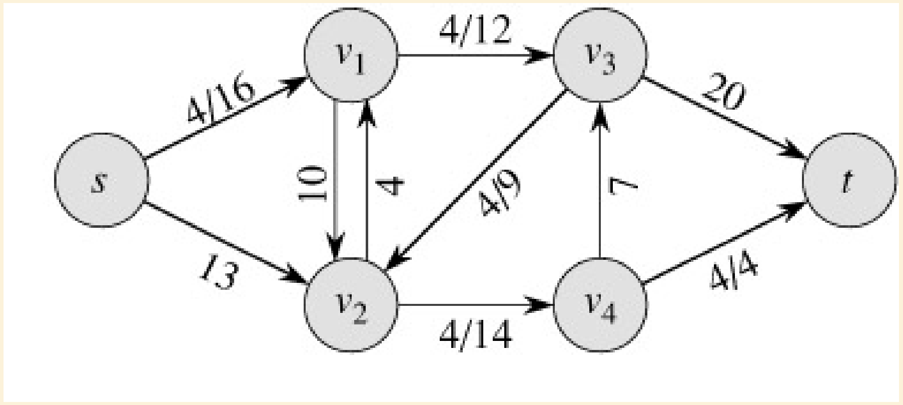


Flow Network

Resulting Flow = 4

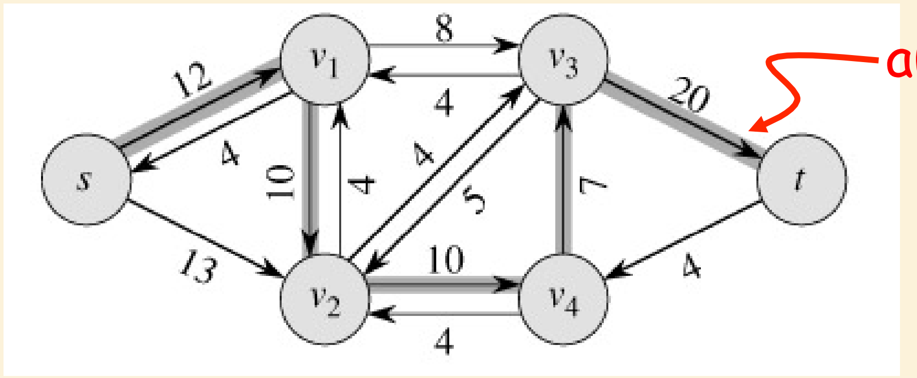
Example

Flow Network



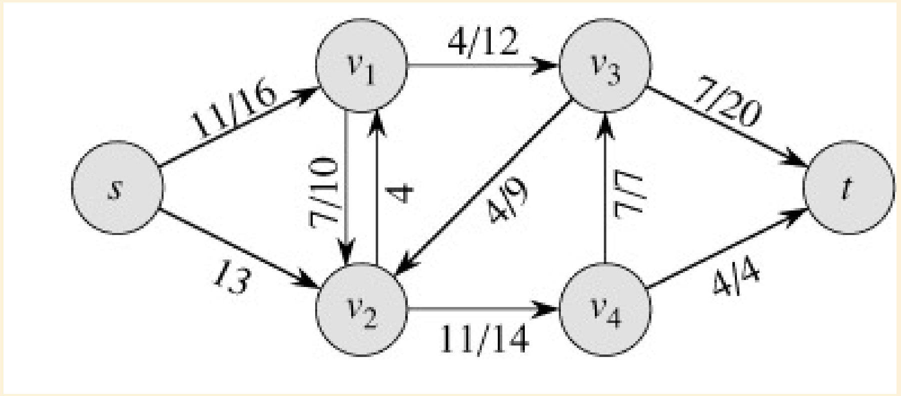
Resulting Flow = 4

Residual Network



augmenting path

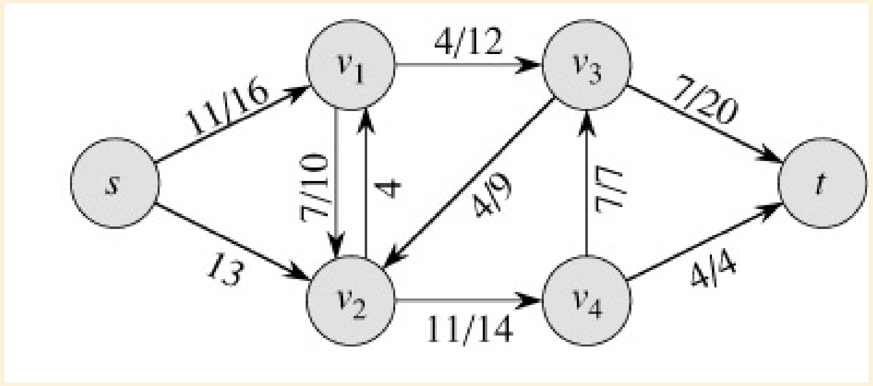
Flow Network



Resulting Flow = 11

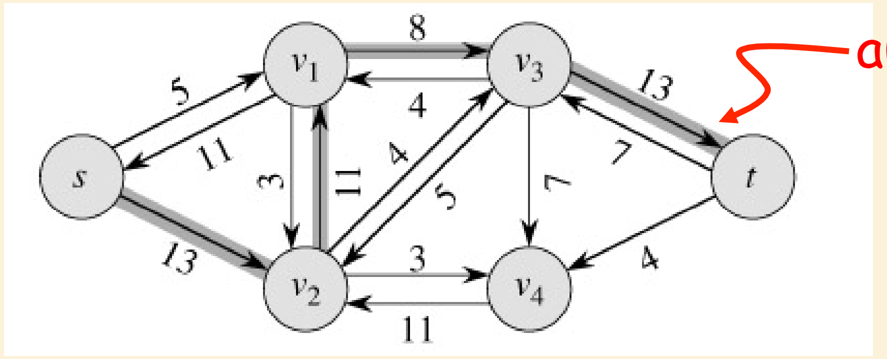
Example

Flow Network



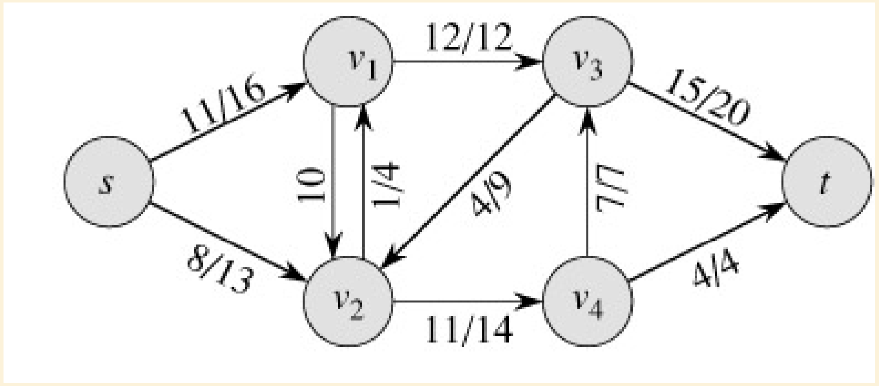
Resulting Flow = 11

Residual Network



augmenting path

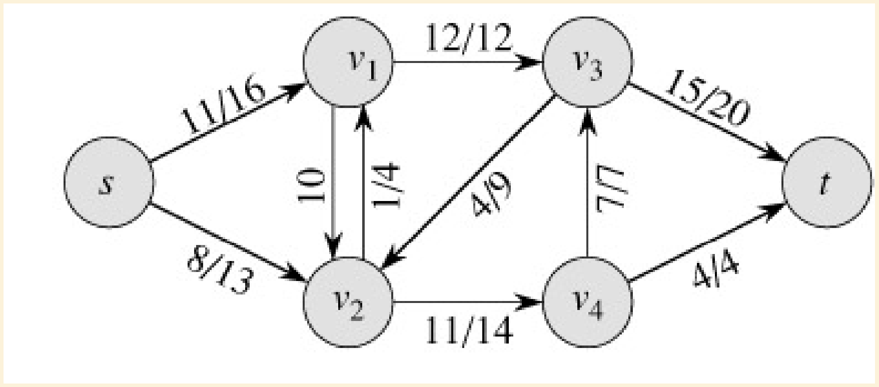
Flow Network



Resulting Flow = 19

Example

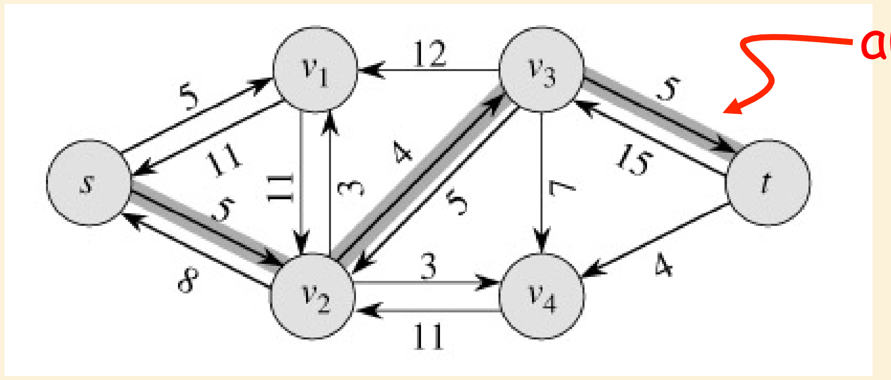
Flow Network



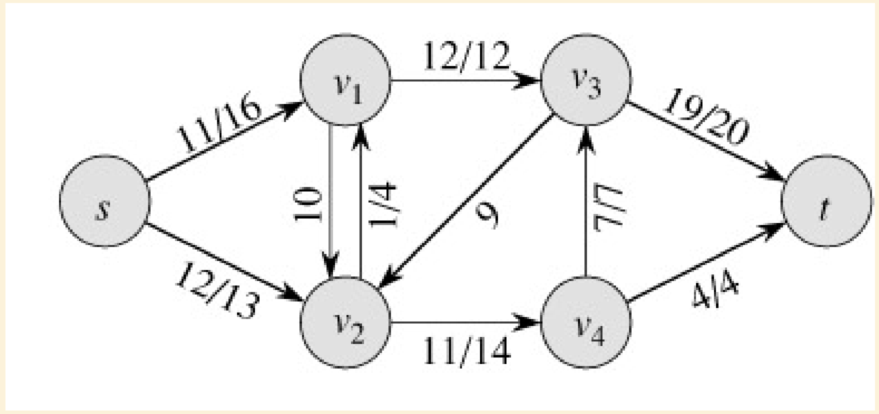
Resulting Flow = 19

Residual Network

augmenting path

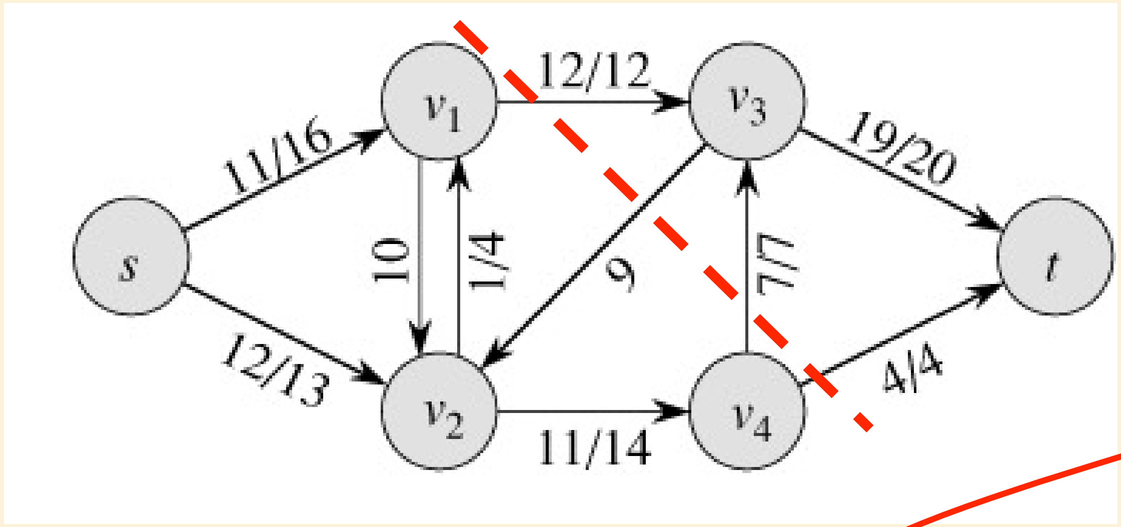


Flow Network



Resulting Flow = 23

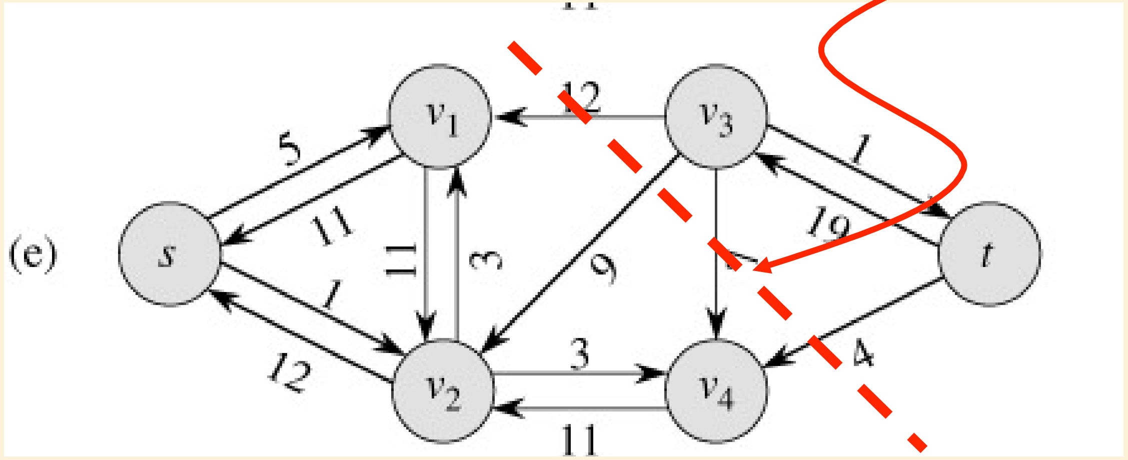
Example



Resulting Flow = 23

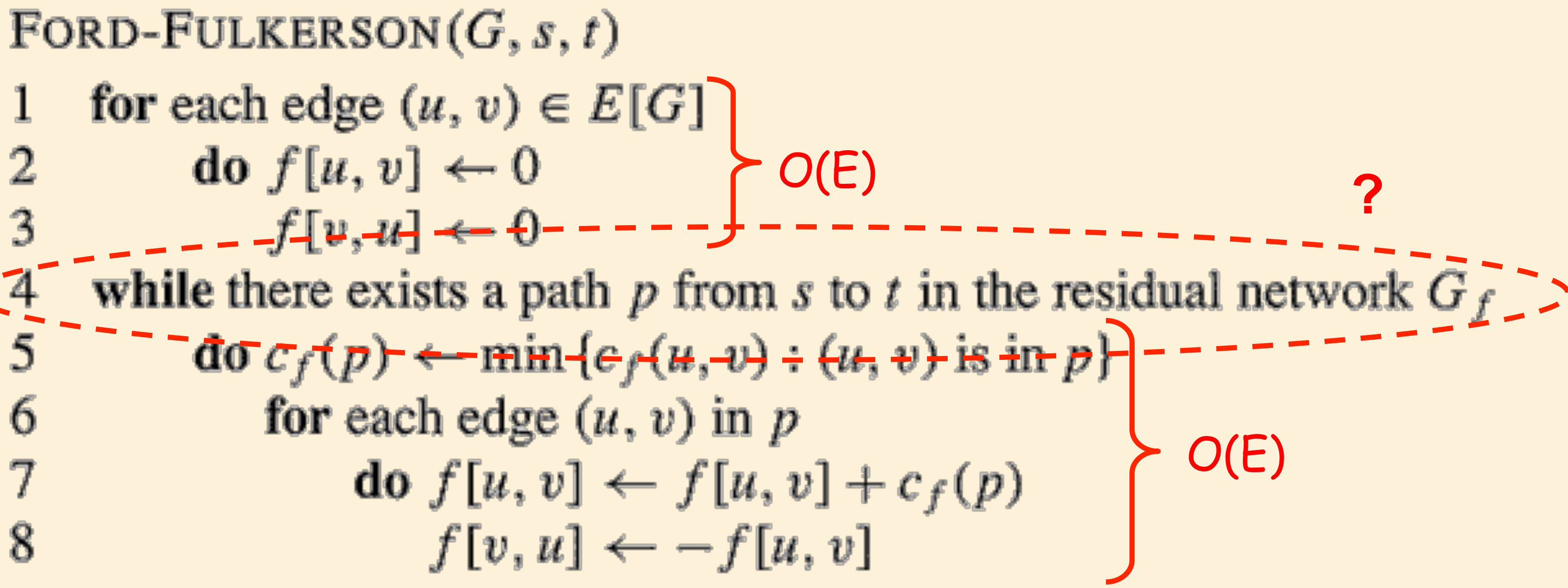
No augmenting path:

Maxflow=23



Residual Network

Analysis



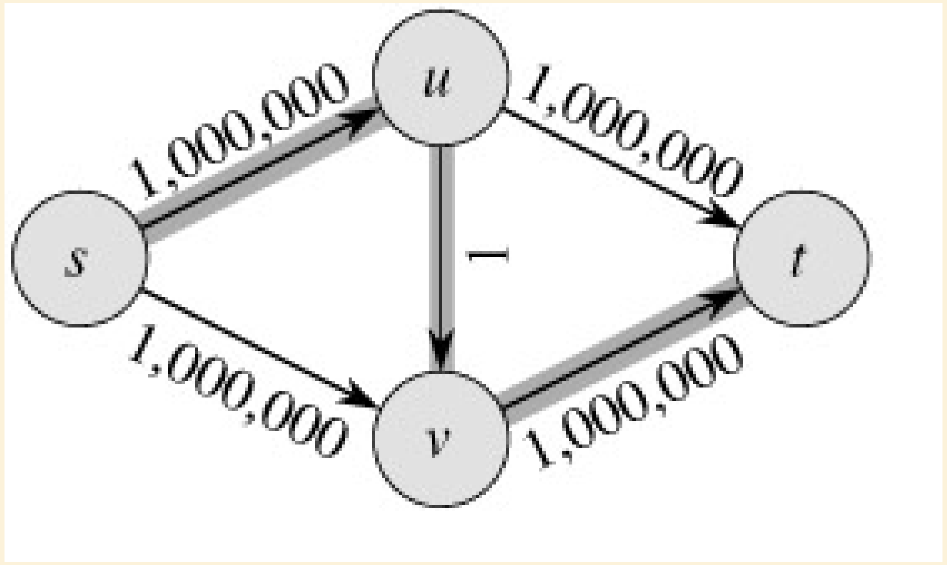
```
FORD-FULKERSON($G, s, t$)
1  for each edge $(u, v) \in E[G]$
2       do $f[u, v] \leftarrow 0$                                   O(E)
3          $f[v, u] \leftarrow 0$
4  while there exists a path $p$ from $s$ to $t$ in the residual network $G_f$      ?
5       do $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v)$ is in $p\}$
6          for each edge $(u, v)$ in $p$
7              do $f[u, v] \leftarrow f[u, v] + c_f(p)$             O(E)
8                 $f[v, u] \leftarrow -f[u, v]$
```

Analysis

- If capacities are all integer, then each augmenting path raises |f| by ≥ 1.
- If max flow is f*, then need ≤ |f*| iterations → time is O(E|f*|).
- Note that this running time is **not polynomial** in input size. It depends on |f*|, which is not a function of |V| or |E|.
- If capacities are rational, can scale them to integers.
- If irrational, FORD-FULKERSON might never terminate!

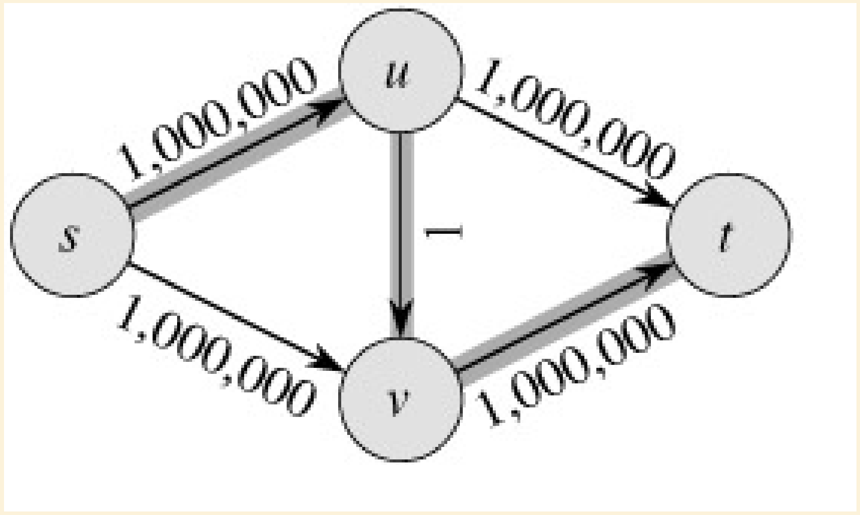
The Basic Ford-Fulkerson Algorithm

- With time O (E |f*|), the algorithm is **not** polynomial.
- This problem is real: Ford-Fulkerson may perform very badly if we are unlucky:

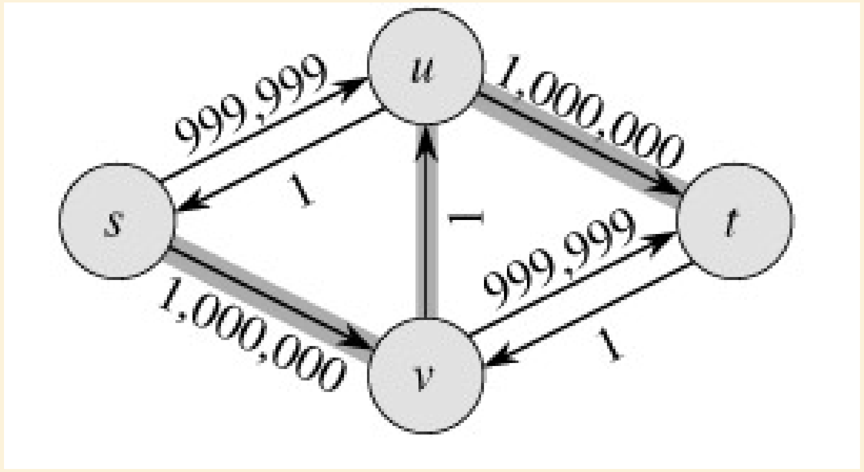


|f*|=2,000,000

Run Ford-Fulkerson on this example

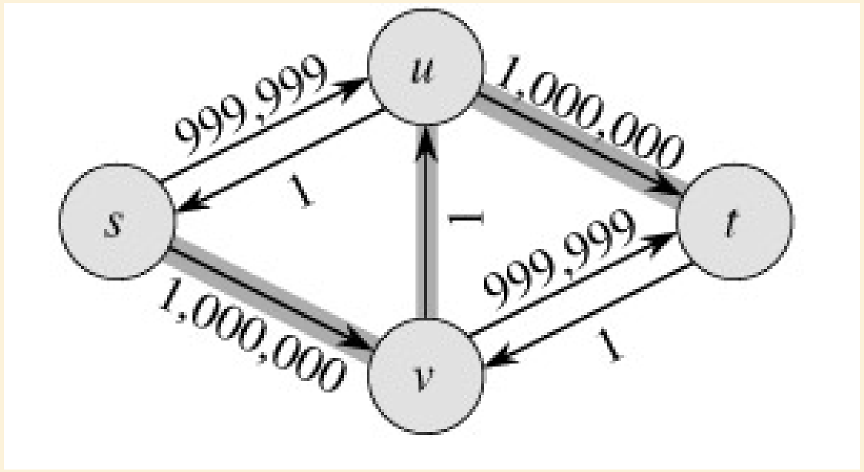


Augmenting Path



Residual Network

Run Ford-Fulkerson on this example



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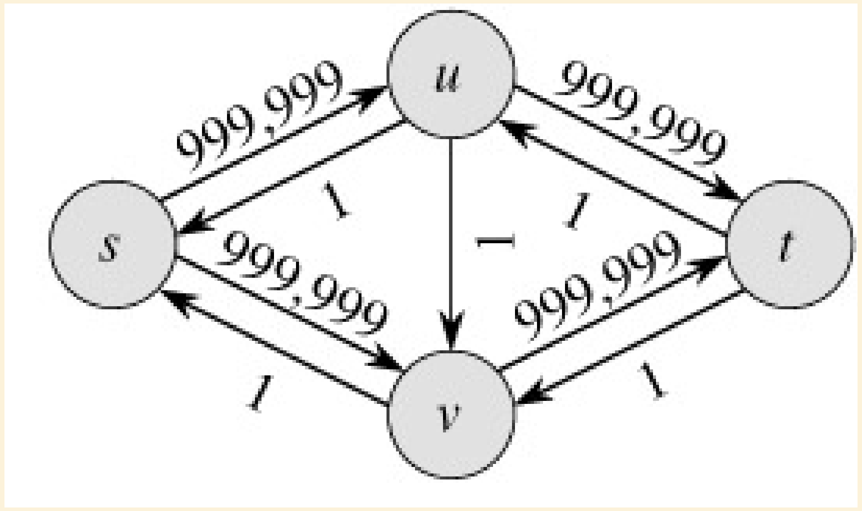

Residual Network

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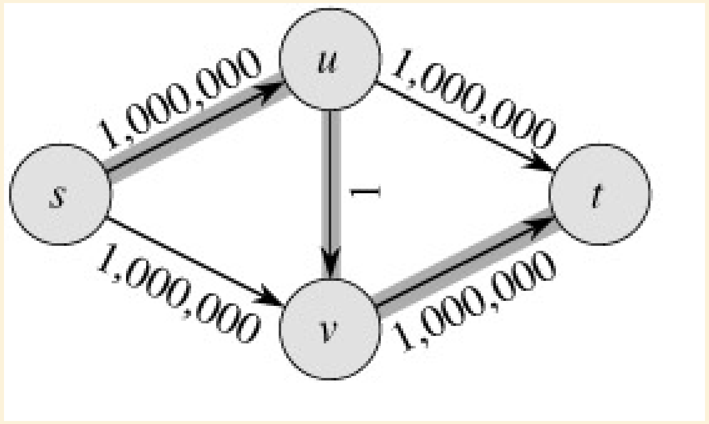
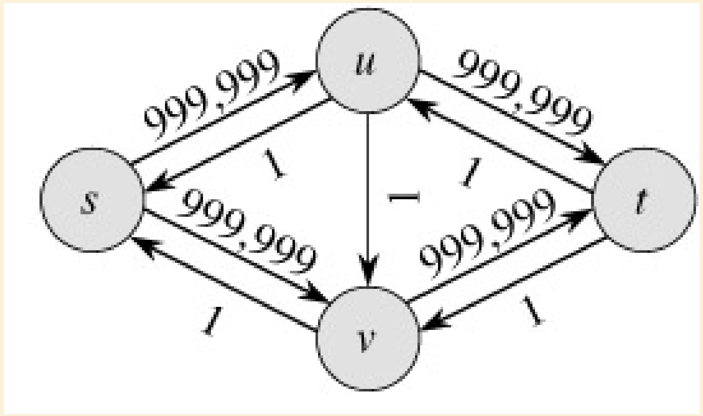
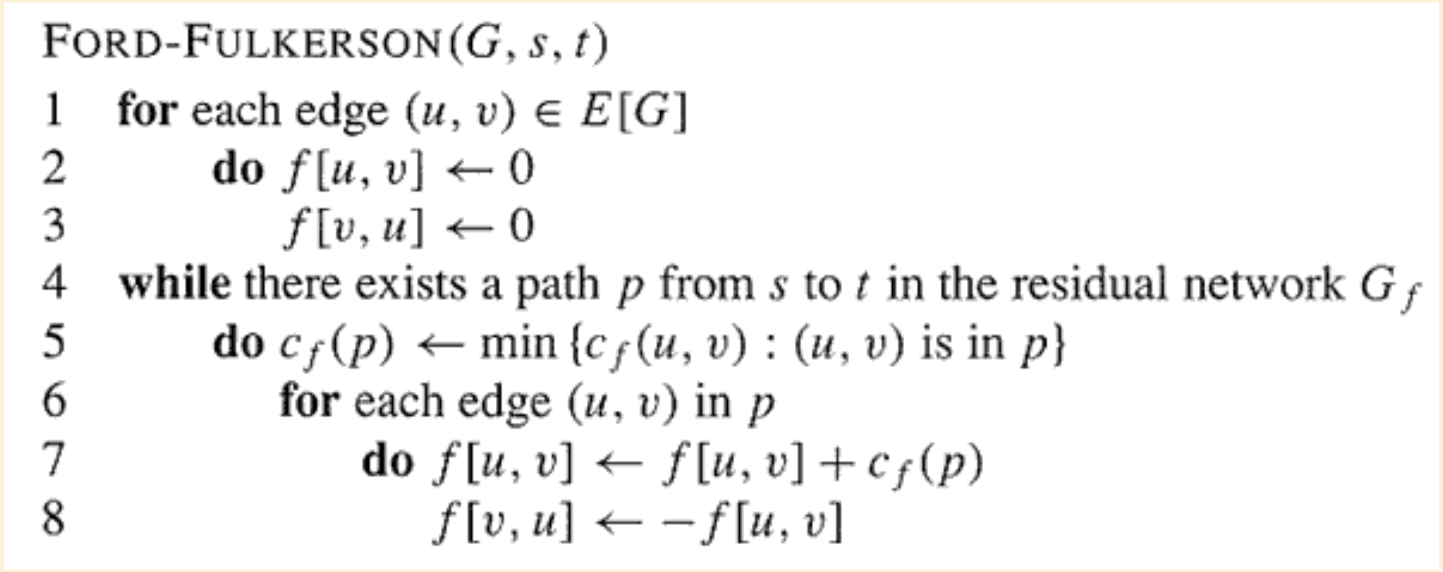




- Repeat 999,999 more times…
- Can we do better than this?

The Edmonds-Karp Algorithm

- A small fix to the Ford-Fulkerson algorithm makes it work in polynomial time.
- Select the augmenting path using **breadth-first search** on residual network.
- The augmenting path *p* is the shortest path from *s* to *t* in the residual network (treating all edge weights as 1).
- Runs in time $O(V E^2)$.



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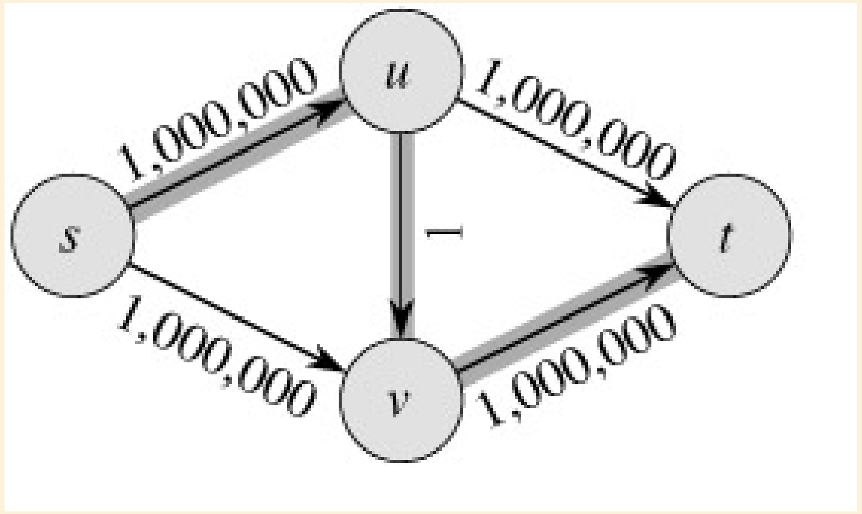
              $f[v, u] \leftarrow -f[u, v]$
```

The Edmonds-Karp Algorithm - example



- The Edmonds-Karp algorithm halts in only 2 iterations on this graph.

Further Improvements

- Push-relabel algorithm ([CLRS, 26.4]) – $O(V^2 E)$.
- The relabel-to-front algorithm ([CLRS, 26.5) – $O(V^3)$.
- (We will not cover these)

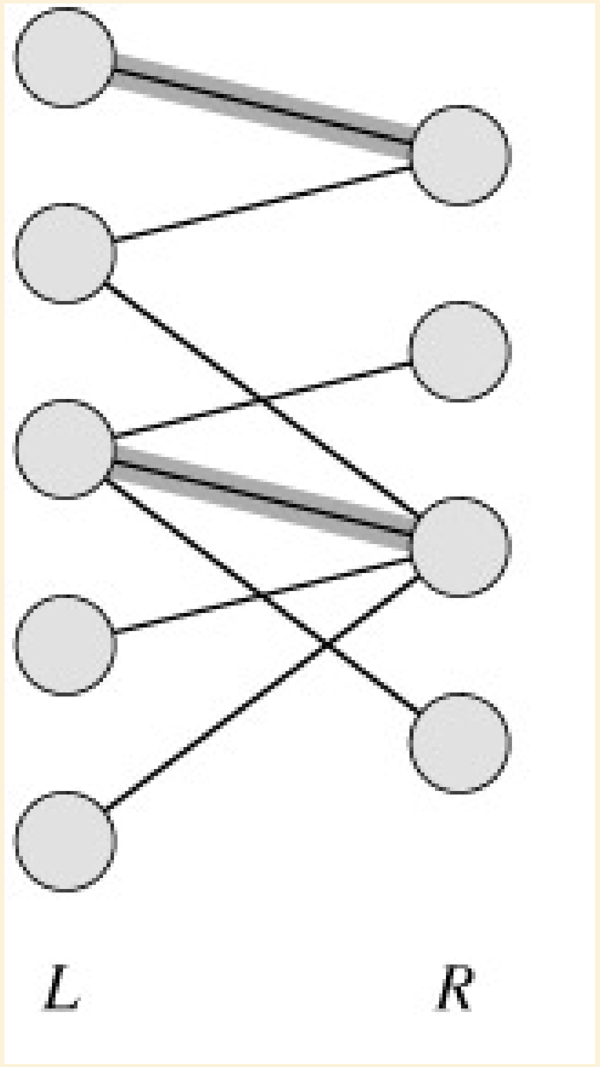
An Application of Max Flow:

Maximum Bipartite Matching

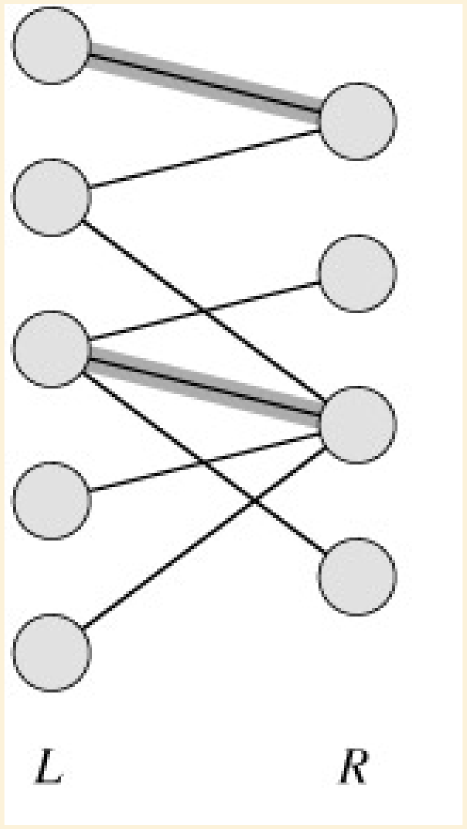
Maximum Bipartite Matching

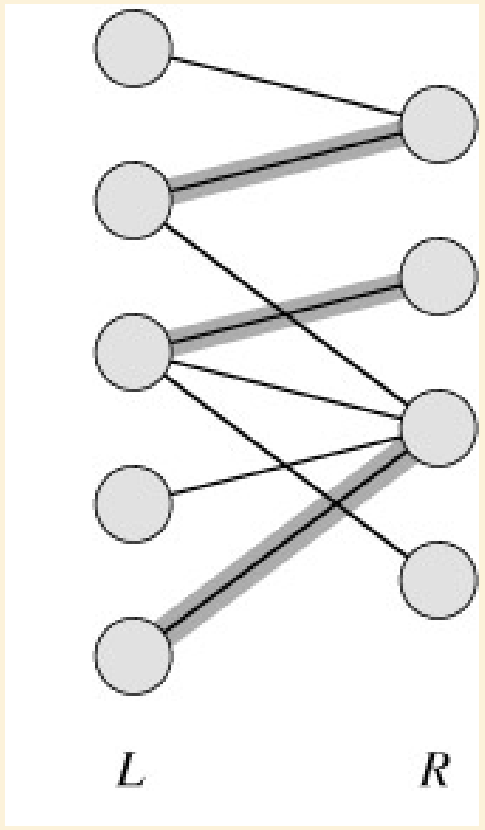
- A bipartite graph is a graph *G=(V,E)* in which *V* can be divided into two parts *L* and *R* such that every edge in *E* is between a vertex in *L* and a vertex in *R*.

- e.g. vertices in *L* represent skilled workers and vertices in *R* represent jobs. An edge connects workers to jobs they can perform.



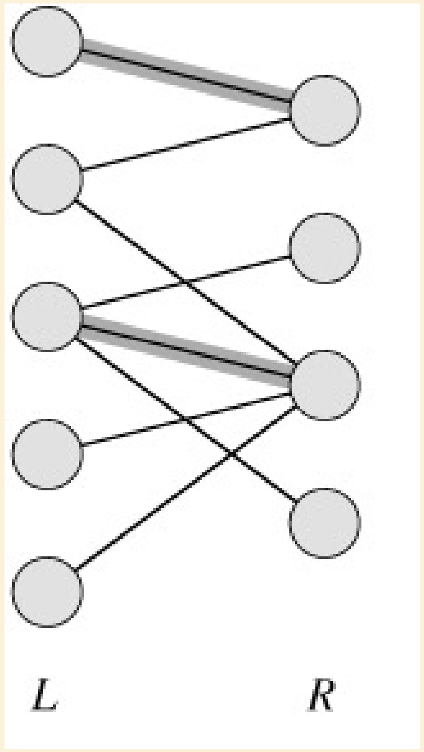
- A matching in a graph is a subset *M* of *E*, such that for all vertices *v* in *V*, at most one edge of *M* is incident on *v*.



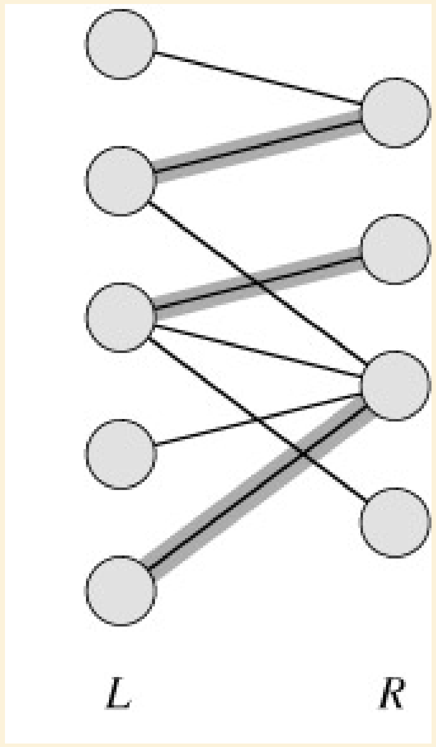


- A **maximum matching** is a matching of maximum cardinality (maximum number of edges).

not maximum

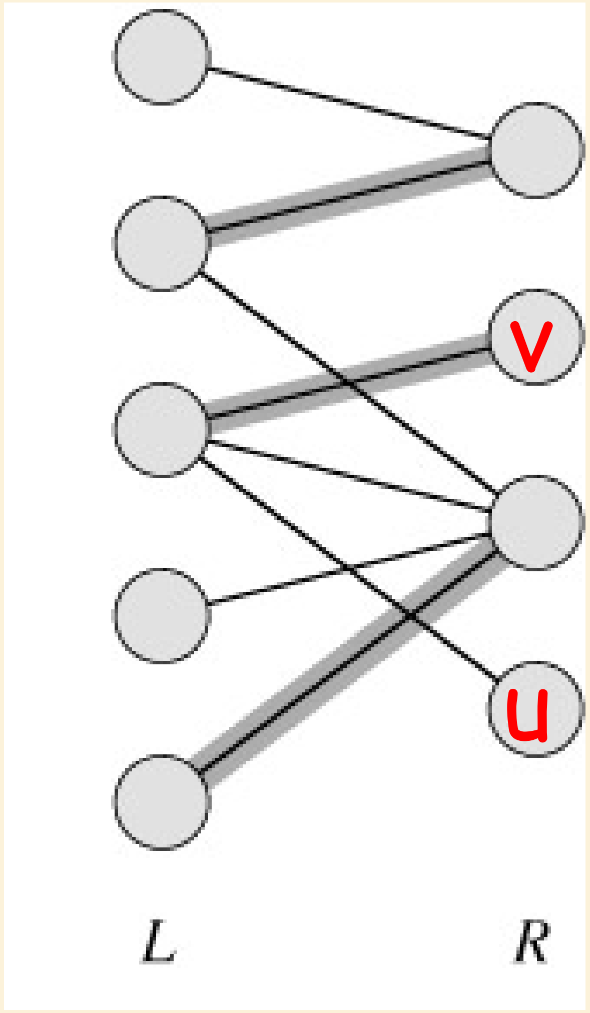


maximum



A Maximum Matching

- No matching of cardinality 4, because only one of v and u can be matched.
- In the workers-jobs example a max-matching provides work for as many people as possible.

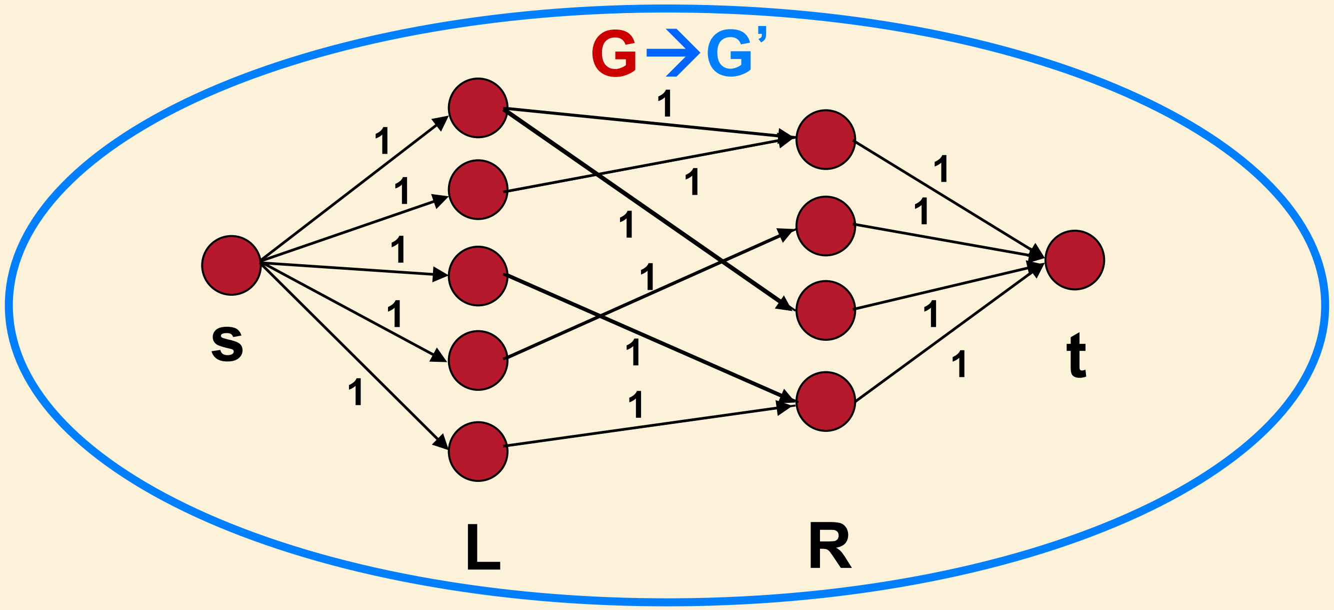


Solving the Maximum Bipartite Matching Problem

- Reduce the maximum bipartite matching problem on graph **G** to the max-flow problem on a corresponding flow network **G'**.
- Solve using Ford-Fulkerson method.

Corresponding Flow Network

- To form the corresponding flow network **G'** of the bipartite graph **G**:
  - Add a source vertex s and edges from s to L.
  - Direct the edges in E from L to R.
  - Add a sink vertex t and edges from R to t.
  - Assign a capacity of 1 to all edges.
- Claim: max-flow in **G'** corresponds to a max-bipartite-matching on **G**.



Solving Bipartite Matching as Max Flow

Let $G = (V, E)$ be a bipartite graph with vertex partition $V = L \cup R$.

Let $G' = (V', E')$ be its corresponding flow network.

If M is a matching in G,

then there is an integer-valued flow f in G' with value $|f| = |M|$.

Conversely, if f is an integer-valued flow in G',

then there is a matching M in G with cardinality $|M| = |f|$.

Thus $\max|M| = \max(\text{integer } |f|)$

Does this mean that max |f| = max |M|?

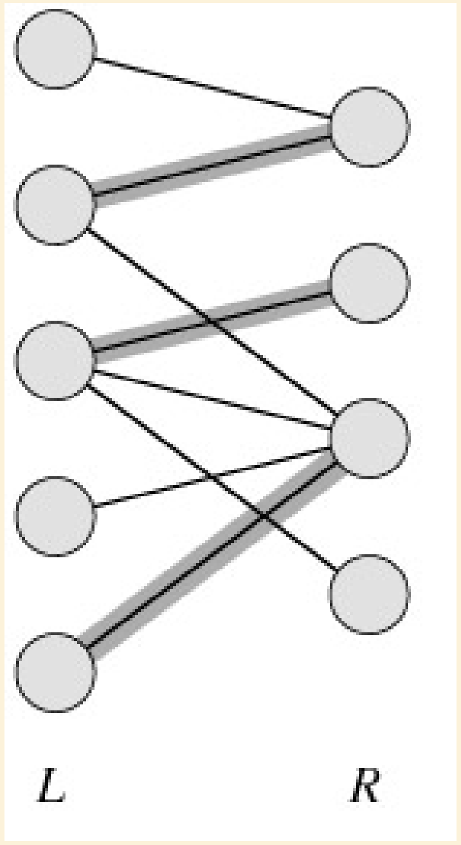
- Problem: we haven't shown that the max flow f(u,v) is necessarily integer-valued.

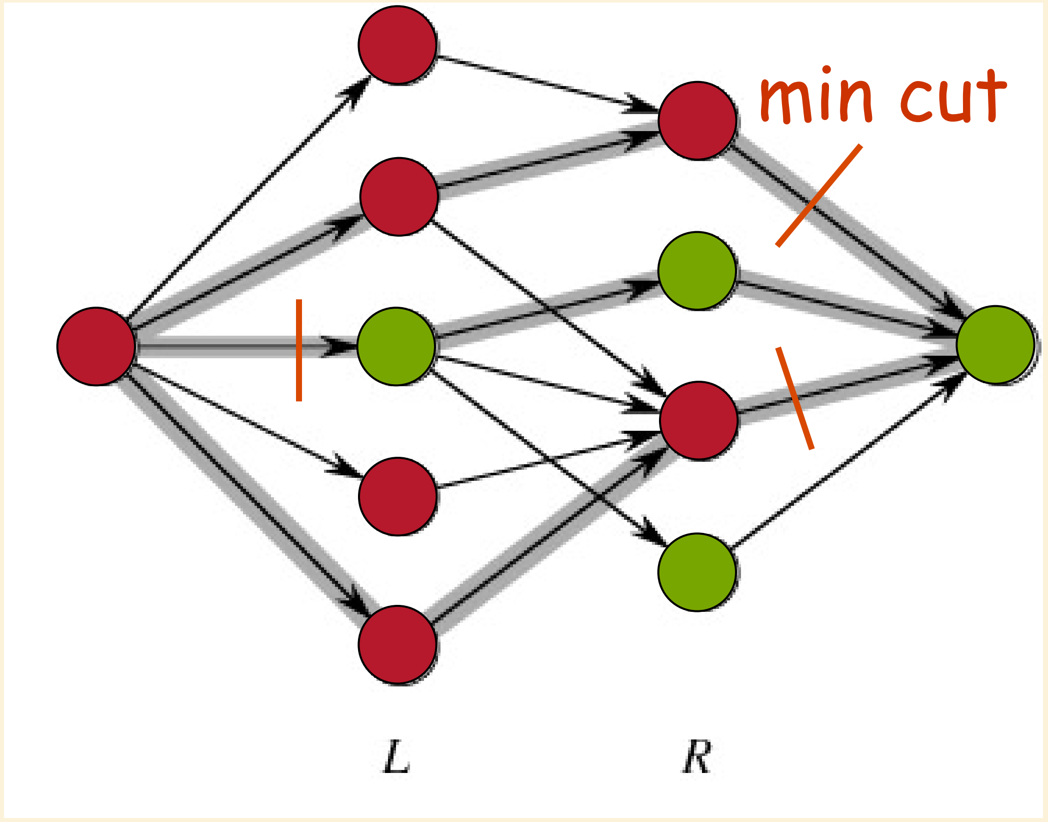
Integrality Theorem

- If the capacity function c takes on only integral values, then:
  1. The maximum flow f produced by the Ford-Fulkerson method has the property that |f| is integer-valued.
  2. For all vertices u and v the value f(u,v) of the flow is an integer.

- So max|M| = max |f|

Example





|M| = 3 ←→ max flow =|f|= 3

Conclusion

- Network flow algorithms allow us to find the maximum bipartite matching fairly easily.

- Similar techniques are applicable in other combinatorial design problems.

Example

- In a department there are n courses and m instructors.
- Every instructor has a list of courses he or she can teach.
- Every instructor can teach at most 3 courses during a year.

- The goal: find an allocation of courses to the instructors subject to these constraints.